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A NOTE ON C^0 TIMOSHENKO BEAM ELEMENTS

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Abstract

A simple analytical description of the problem of shear 'locking' of a linear four-degree-of-freedom C^0 Timoshenko beam element is presented. The description of locking is based on comparison of the element stiffness matrix with the corresponding matrix for a 'standard' non-locking element and on comparison of the analytical solutions of the discretized equations with the exact solution. It is shown that the problem of locking can be avoided by use of reduced bending and shear rigidities. The standard remedies for shear locking involving reduced integration and higher order elements are also discussed and the relation between different strategies to avoid ill-conditioning for extremely slender beams is examined.

Introduction

Timoshenko beam elements with C^0-continuity, i.e. elements in which the continuity of slope is not assured, have been used extensively to illustrate the effects of ill-conditioning, locking and of selective reduced integra-

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tion (1, 2, 3, 4, 5). The typical finding is that, for certain types of C^0 elements, the assembled model of the beam tends to 'lock', i.e. to become extremely stiff, when the slendemess of the beam is increased while the number of elements is kept fixed. The standard approach to meliorate this situation has involved the use of selective reduced integration to calculate the stiffness terms associated with shear deformation. Hughes *et al.* (2) have presented a 4-degree-of-freedom reduced integration C^0 element which exhibits excellent behavior over a wide range of the slendemess ratio. This element fails only when the slendemess ratio exceeds a high value which depends on the word length of the particular computer used. This problem is corrected in turn by dividing the shear stiffness by a factor proportional to the square of the slendemess ratio of the element (2), or, by introducing a reduced shear rigidity (6).

Fried (1) considered a 5-degree-of-freedom C^0 element for a beam and found that the stiffness matrix became violently ill-conditioned as the depth of the beam was reduced. He found that the ill-conditioning could be removed if a fictitious beam depth proportional to the length of the element was used to calculate the relative contribution of the shear energy. He also suggested that an optimum value of the constant of proportionality could be found by numerical experimentation. Fried (1973) argues that for thin beams the discretization error is much larger than the shear correction and does not warrant the use of the exact value for the depth of the beam. He proposed instead the use of a fictitious depth such that the discretization error and the shear correction are balanced.

The first objective of this note is to examine in an elementary and unified fashion the problem of locking of C^0 beam elements as well as the proposed remedies for this problem. A second objective is to study the connection between the 4-degree-of-freedom reduced integration C^0 element considered by Hughes *et al.* (2) and the 5-degree-of-freedom full integration C^0 element studied by Fried (1). Both of these elements are non-locking but suffer from ill-conditioning at high slendemess ratios. The final objective is to examine the relation between the remedies for the ill-conditioning proposed by Fried (1) and Hughes *et al.* (2) and the reduction of the shear rigidity advocated by MacNeal (6). The approach taken here is based on comparing the element stiffness matrices for different C^0 elements with a 'standard' stiffness matrix corresponding to the exact force-displacement relation for a Timoshenko beam sujected to end loads. Additional insight is obtained by comparison of analytical solutions fot the deflection and rotation of cantilever Timoshenko beams discretized by various types of C^0 element with the exact solution for the non-discretized beam.

BASIC EQUATIONS

The element stiffness matrices considered here are derived from the following expression for the strain energy in each element:

$$U_e = \frac{EI}{2}\left[\int_0^h \left(\frac{d\theta}{dx}\right)^2 dx + \kappa\frac{GA}{EI}\int_0^h \left(\frac{dw}{dx} - \theta\right)^2 dx\right] \quad (1)$$

where w(x) is the lateral displacement of the centerline, θ(x) is the rotation of the cross-section, E is Young's modulus, G is the shear modulus, κ is the shear correction factor, A is the area of the cross-section, I is the moment of inertia of the cross-section with respect to the neutral axis, h is the length of the element and x is the axial coordinate. The first term on the right-hand side of Eq. (1) is, of course, the bending energy and the second is the shear energy. It is convenient to introduce the normalized variables $\xi = x/h$ and $\hat{\theta} = h\theta$. With these substitutions, the strain energy can be written in the form

$$U_e = \frac{1}{2}\left(\frac{EI}{h^3}\right)\left[\int_0^1 \left(\frac{d\hat{\theta}}{d\xi}\right)^2 d\xi + 6\gamma_e^2\int_0^1 \left(\frac{dw}{d\xi} - \hat{\theta}\right)^2 d\xi\right] \quad (2)$$

where

$$\gamma_e = \left[\frac{\kappa GAh^2}{6\,EI}\right]^{1/2} = \sqrt{\frac{\kappa}{1+\nu}}\left(\frac{h}{r\sqrt{12}}\right) \quad (3)$$

in which $r = \sqrt{\frac{I}{A}}$ is the radius of gyration of the cross-section. The relation $E = 2(1+\nu)G$ where ν is Poisson's ratio, has been used in Eq. (3). The parameter γ_e represents a normalized slendemess ratio of the element. For a rectangular cross-section of depth t, $r = t/\sqrt{12}$ and this parameter takes the value $\nu_e = h/t$ if $\kappa = 1$ and $\nu = 0$.

Eqs. (1) and (2) suggest that the element stiffness matrix $[k_e]$ can be written in the form

$$[k_e] = \left(\frac{EI}{h^3}\right)([k_b] + [k_s]) \quad (4)$$

where $[k_b]$ and $[k_s]$ are normalized stiffness matrices associated with bending and shear deformations, respectively.

As a reference for later comparisons, we consider an element obtained by use of shape functions corresponding to the exact solution for a Timoshenko beam segment of lenght h subjected to end forces and moments. The exact solutions for the lateral deflection and normalized rotation are given by

$$(1+2\gamma_e^{-2})\, w\,(h\xi) = (1-\xi)\,(1+\xi-2\xi^2+2\gamma_e^{-2})\, w_1 + \xi(1-\xi)$$
$$(1-\xi+\gamma_e^{-2})\,\hat{\theta}_1 + \xi(3\xi-2\xi^2+2\gamma_e^{-2})\, w_2 - \xi(1-\xi)\,(\xi+\gamma_e^{-2})\,\hat{\theta}_2 \tag{5a}$$

$$(1+2\gamma_e^{-2})\,\hat{\theta}\,(h\xi) = -6\xi(1-\xi)w_1 + (1-\xi)\,(1-3\xi+2\gamma_e^{-2})\hat{\theta}_1$$
$$+\,6\xi\,(1-\xi)w_2 + \xi\,(-2+3\xi+2\gamma_e^{-2})\,\hat{\theta}_2 \tag{5b}$$

where w_1, w_2 are the end deflections and $\hat{\theta}_1, \hat{\theta}_2$ the normalized end rotations. Substitution from Eqs. (5a) and (5b) into Eq. (2) leads to

$$[k_b] = \frac{2}{(1+2\gamma_e^{-2})^2}\begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & 2(1+\gamma_e^{-2}+\gamma_4^{-4}) & -3 & (1-2\gamma_e^{-2}-2\gamma_e^{-4}) \\ -6 & -3 & 6 & -3 \\ 3 & (1-2\gamma_e^{-2}-2\gamma_e^{-4}) & -3 & 2(1+\gamma_e^{-2}+\gamma_e^{-4}) \end{bmatrix} \tag{6}$$

$$[k_s] = \frac{4\gamma_e^{-2}}{(1+2\,\gamma_e^{-2})^2}\begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & 3/2 & -3 & 3/2 \\ -6 & -3 & 6 & -3 \\ 3 & 3/2 & -3 & 3/2 \end{bmatrix} \tag{7}$$

in which the degrees-of-freedom are ordered in the form w_1, $\hat{\theta}_1$, w_2 and $\hat{\theta}_2$. The total element stiffness matrix is given by

$$[k_e] = \frac{2}{(1+2\gamma_e^{-2})^2}\left(\frac{EI}{h^3}\right)\begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & (2+\gamma_e^{-2}) & -3 & (1+\gamma_e^{-2}) \\ -6 & -3 & 6 & -3 \\ 3 & (1-\gamma_e^{-2}) & -3 & (2+\gamma_e^{-2}) \end{bmatrix} \tag{8}$$

As γ_e tends infinity, the matrices $[k_e]$ and $(EI/h^3)[k_b]$ tend to the 'standard' stiffness matrix for a Bernoulli beam and the matrix $[k_s]$ tends to zero. It should be noted that the rank of the matrix $[k_s]$ is one. In what follows, the matrices $[k_b]$, $[k_s]$ and $[k_e]$ are referred to as the 'standard' bending, shear and total stiffness matrices for a Timoshenko beam element.

For future reference, we also note that the deflection and rotation of a uniform cantilever Timoshenko beam fixed at the end x = 0 and subjected to a force P at the end x = l are given by

$$w(x) = \frac{Pl^3}{3El} \left\{ \left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(1 - \frac{x}{l}\right)\left(\frac{x}{l}\right)^2 + \frac{1}{2\gamma^2}\left(\frac{x}{l}\right) \right\} \quad (9a)$$

$$\theta(x) = \frac{Pl^2}{2El} \left\{ \left(\frac{x}{l}\right)^2 + 2\left(1 - \frac{x}{l}\right)\frac{x}{l} \right\} \quad (9b)$$

where

$$\gamma = \left[\frac{KGAl^2}{6El}\right]^{1/2} = \sqrt{\frac{\kappa}{1+\nu}}\left(\frac{l}{r\sqrt{12}}\right) \quad (10)$$

is a normalized slenderness ratio for the full beam. Finally, the end displacement and rotation are given by

$$w(l) = \left(\frac{Pl^3}{3El}\right)\left(1 + \frac{1}{2\gamma^2}\right) = \Delta(\gamma) \quad (11a)$$

$$\theta(l) = \left(\frac{Pl^2}{2\ El}\right) \quad (11b)$$

where $\Delta(\infty) = (Pl^3/3El)$ corresponds to the end displacement for the Bemoulli beam.

A C^0 LOCKING ELEMENT

As an example of a locking C^0-element, we consider, after Hughes *et al.* (2), a 4-degree-of-freedom element in which the lateral displacement and rotation are described by the linear shape functions

$$w(\xi) = (1 - \xi)\, w_1 + \xi\, w_2 \quad (12a)$$

$$\hat{\theta}(\xi) = (1 - \xi)\,\hat{\theta}_1 + \xi\,\hat{\theta}_2 \tag{12b}$$

and in which the stiffness matrices are calculated by exact integration of the strain energy. Substitution from Eqs. (12a) and (12b) into Eq. (2), followed by exact integration, leads to

$$[k'_b] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{13}$$

and

$$[k'_s] = \gamma_e^2 \begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & 2 & -3 & 1 \\ -6 & -3 & 6 & -3 \\ 3 & 1 & -3 & 2 \end{bmatrix} \tag{14}$$

in which the degrees-of-freedom are again ordered in the form $w_1, \hat{\theta}_1, w_2, \hat{\theta}_2$. It is apparent that the rank of $[k'_b]$ is one while that of $[k'_s]$ is two. The total stiffness matrix for the element is, in this case,

$$[k'_e] = \gamma_e^2 \left(\frac{EI}{h^3}\right) \begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & (2+\gamma_e^{-2}) & -3 & (1-\gamma_e^{-2}) \\ -6 & -3 & 6 & -3 \\ 3 & (1-\gamma_e^{-2}) & -3 & (2+\gamma_e^{-2}) \end{bmatrix} \tag{15}$$

Inspection of Eq. (15) reveals that, as $\gamma_e \to \infty$, the matrix $[k'_e]$ is proportional to the stiffness matrix for the 'standard' beam element for a Bernoulli beam and that the factor of proportionality is $(\gamma_e^2/2)$. For a beam modelled by a number of equal elements, the deflection for the C^0 model, in the limit as $\gamma_e \to \infty$, will be equal to that for the Bernoulli beam multiplied by $(2/\gamma_e^2)$, and, consequently, will be much smaller than the solution for the Bernoulli beam.

A more detailed description of the locking effect can be obtained by noting that the stiffness matrix $[k'_e]$ for the C^0-element given by Eq. (15) is exactly proportional to the stiffness matrix $[k_e]$ given by Eq. (8) for the 'standard' Timoshenko beam element. The factor of proportionaly is 1 + $(\gamma_e^2/2)$. Consider now a uniform cantilever beam fixed at $x = 0$ and subjected to an end load P at x = 1. Let the beam be discretized into N equal elements of length h (1 = Nh). In this case, the 'standard' element. characterized by Eqs. (5a), (5b) and (8) leads to the exact solution. On the basis of the proportionality of the stiffness matrices $[k_e]$ and $[k_e']$, it follows that the nodal deflection $w'(x)$ and nodal rotation $\theta'(x)$ (x = nh, n = 0, N) obtained by use of the C^0-element are

$$w'(x) = \frac{1}{1+(\gamma_e^2/2)} w(x), \theta'(x) = \frac{1}{1+(\gamma_e^2/2)} \theta(x), (x = nh, n = 0, N) \tag{16}$$

where w(x) and θ(x) are the exact displacement and rotation given by Eqs. (9a) and (9b). In particular, if $\Delta'(\gamma,N) = w'(1)$ denotes the end deflection for the C^0-element and $\Delta(\gamma)$ and $\Delta(\infty) = \frac{Pl^3}{3EI}$ denote the exact deflections for the Timoshenko and Bernoulli beams, respectively, then

$$\frac{\Delta'(\gamma, N)}{\Delta(\gamma)} = \frac{1}{1+\left(\frac{\gamma_e^2}{2}\right)} = \frac{1}{1+\left(\frac{\gamma^2}{2N^2}\right)} \tag{17}$$

$$\frac{\Delta'(\gamma, N)}{\Delta(\infty)} = \frac{1+\left(\frac{1}{2\gamma^2}\right)}{1+\left(\frac{\gamma^2}{2N^2}\right)} \tag{18}$$

where $\gamma = N\gamma_e$. Eqs. (17) and (18) show that if N is kept fixed, then $\Delta'(\gamma,N)/\Delta(\gamma)$ and $\Delta'(\gamma,N)/\Delta(\infty)$ tend to zero as γ tends to infinity. This locking behaviour is ullustrated in Fig. 1 where the ratio $\Delta'(\gamma,N)/\Delta(\gamma)$ is shown versus γ for different values of N. A comparison of results based on Eq. (17) with numerical results obtained by Hughes *et al.* (2) is presented in the Appendix.

Eq. (17) indicates that the deflection for the C^0-element will approximate the exact solution of the Timoshenko beam only if the parameter $\gamma_e = \gamma N$ is kept small. The results in Table 1 indicate that the length h of the element would have to be less than one quarter of the height of a beam of rectangular cross-section ($r = t/\sqrt{12}, \nu = 1/3, \kappa = 5/6$) and γ_e should be less than 0.20 for the error to be less than 2 percent. This requirement implies an intolerably large number of elements for slender beams ($N \approx 5\gamma$ for 2 percent error).

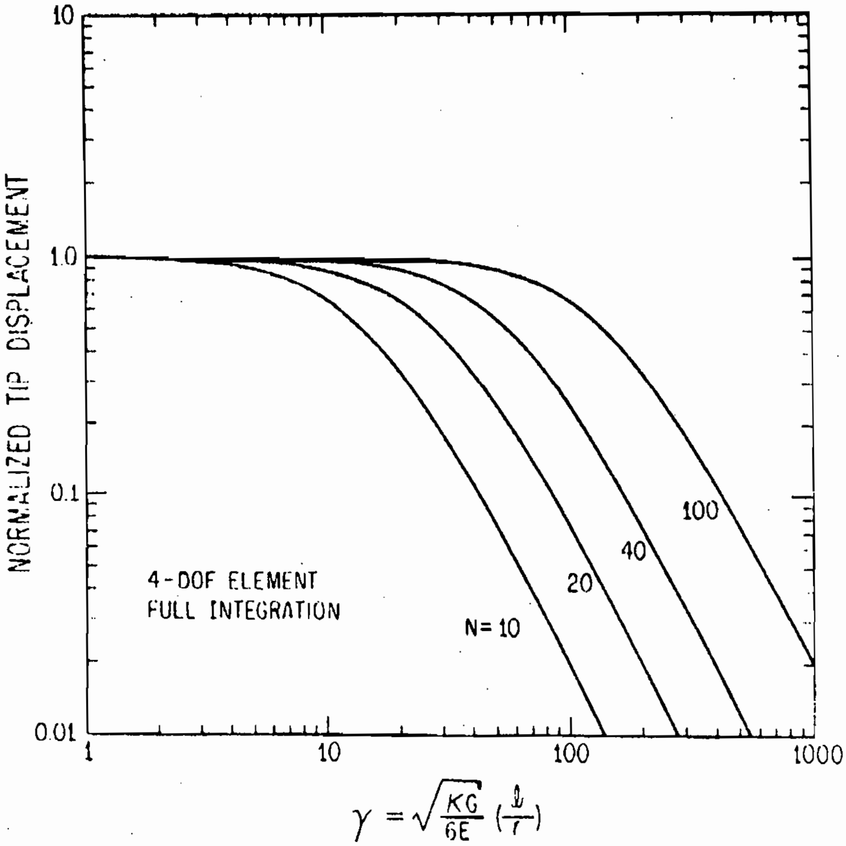

Fig. 1. Tip deflection of cantilever beam modelled by N equal four-degree-of-freedom full integration elements plotted versus the normalized slenderness ratio of the beam γ. The tip deflection is normalized by the exact result for a Timoshenko beam.

TABLE 1

VALUES OF $\Delta'(\gamma N)/\Delta(\gamma)$ AND γ, VERSUS h/t FOR A BEAM WITH RECTANGULAR CROSS-SECTION

$r = t/\sqrt{12}$, $\nu = 1/3$, $\kappa = 5/6$

(h/t)	1,0	0,5	0,25	0,125
γ_e	0,791	0,395	0,198	0,099
$\Delta'(\gamma N)/\Delta(\gamma)$	0,762	0,928	0,981	0,995

From Eq. (18) it is found that the end deflection for the C^0-element is equal to the exact Bernoulli result if $N = \gamma^2$. If an error η with respect to the Bernoulli result is considered admissible then the required number of elements is $N \sim \gamma \sqrt{(1 - \eta)/2\eta}$ for γ large. Thus, for slender beams, a large number of elements is also requerired to match the Bernoulli result.

The locking of the linear 4-degree-of-freedom C^0-element just considered can be eliminated by use of a higher order element (1) or by use of selective reduced integration (2) as discussed in the following section. A simple alternative remedy, which apparently has not been considered in the past, consists of taking advantage of the proportionality of the stiffness matrix $[k_e']$ for the linear C^0-element and the stiffness matrix $[k_e]$ for the 'standard' Timoshenko beam element. If the bending EI and shear κGA rigidities of the C^0-element are replaced by the effective values

$$(EI)_T = \frac{EI}{1 + \frac{\gamma_e^2}{2}} \tag{19}$$

and

$$(\kappa GA)_T = \frac{\kappa GA}{1 + \frac{\gamma_e^2}{2}} \tag{20}$$

then, the normalized element aspect ratio $(\gamma_e)_T^2 = (\kappa GAh^2/6EI)_T = \gamma_e^2$ remains unchanged and the stiffness matrix $[k_e']$ for the modified C^0-element [Eq. (15)] becomes identical to the stiffness matrix $[k_e]$ for the standard Tismoshenko beam element [Eq. (8)]. Since both EI and κGA are affected by the same factor the suggested modification entails only the division of the total element stiffness matrix by

$$\left(1 + \frac{\gamma_e^2}{2}\right)$$

The resulting modified C^0-element leads to exact nodal solutions for loads applied to the nodes. In particular, for the cantilever beam subjected to an end load P considered as example, an exact nodal solution is obtained independently of the number of elements used. In the limiting case of $\gamma_e \rightarrow \infty$, then

$$\gamma_e^2 \left(\frac{EI}{h^3}\right)_T \rightarrow 2 \left(\frac{EI}{h^3}\right)$$

and the matrix $[k'_e]$ tends to the standard form for the Bernoulli beam.

NON-LOCKING C^0-ELEMENTS

Following Hughes *et al.* (2) we consider a 4-degree-of-freedom C^0-element in which the lateral displacements and rotations are given by independent linear shape functions as in Eqs. (12a) and (12b). In this case, the strain energy is calculated by one-point Gaussian quadrature leading to

$$[k_b''] = [k'_b] \tag{21}$$

$$[k_s''] = \gamma_e^2 \begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & 3/2 & -3 & 3/2 \\ -6 & -3 & 6 & -3 \\ 3 & 3/2 & -3 & 3/2 \end{bmatrix} \tag{22}$$

$$[k_e''] = \gamma_e^2 \left(\frac{EI}{h^3}\right) \begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & \left(\frac{3}{2} + \gamma_e^{-2}\right) & -3 & \left(\frac{3}{2} - \gamma_e^{-2}\right) \\ -6 & -3 & 6 & -3 \\ 3 & \left(\frac{3}{2} - \gamma_e^{-2}\right) & -3 & \left(\frac{3}{2} + \gamma_e^{-2}\right) \end{bmatrix} \tag{23}$$

where the degrees-of-freedom are w_1, $\hat{\theta}_1$, w_2 and $\hat{\theta}_2$. The matrix $[k_s'']$ is of rank one and it is proportional to the matrix $[k_s]$ given by Eq. (7) for the 'standard' Timoshenko beam element. It must be noted however that $[k_s]$ tends to zero as $\gamma_e \to \infty$ while $[k_s'']$ tends to infinity. Another observation is that the total element stiffness matrix $[k_e'']$ is not proportional to the stiffness matrix for the 'standard' Timoshenko beam element nor tends to the 'standard' stiffness matrix for a Bernoulli beam as $\gamma_e \to \infty$.

Additional insight into the characteristics of the four-degree-of-freedom reduced integration element can be obtained by considering the analytical solution for the deflection of a uniform cantilever Timoshenko beam of lenght l discretized into N finite elements of equal length h (l = Nh). Treating the global equilibrium equation as a difference equation with appropriate end conditions leads to the solution

$$w''(x) = \frac{Pl^3}{3EI}\left\{\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left[1 - \left(\frac{x}{l}\right)\right]\left(\frac{x}{l}\right)^2 + \frac{1}{2}\left(\frac{1}{\gamma^2} - \frac{1}{2N^2}\right)\left(\frac{x}{l}\right)\right\} \tag{24a}$$

$$\theta''(x) = \frac{Pl^2}{2EI}\left\{\left(\frac{x}{l}\right)^2 + 2\left[1-\left(\frac{x}{l}\right)\right]\left(\frac{x}{l}\right)\right\} \tag{24b}$$

for x = nh (n = 0,N). The rotation coincides with the exact solution given by Eq. (9b) and the lateral displacement differs from the exact solution [Eq. (9a)] only by the term $-Pl^3/3EI)(x/l)(1/2N)^2$ which tends to zero as $N \to \infty$. The ratio of the tip deflection $\Delta''(\gamma, N) = w''(l)$ to the exact tip deflection for a Timoshenko beam is given by

$$\frac{\Delta''(\gamma, N)}{\Delta(\gamma)} = 1 - \frac{\left(\frac{1}{2N}\right)^2}{1 + \frac{1}{2\gamma^2}} \tag{25}$$

The values for this ratio for $\gamma^2 = 10$ and N = 1, 2, 4, 8 and 16 are 0.762, 0.940, 0.985, 0.996 and 0.999, respectively. The corresponding values for $\gamma^2 = 10^6$ are 0.750, 0.938, 0.984, 0.996 and 0.999. All of these values coincide exactly with the numerical results presented by Hughes *et al.* (2).

The test computations conducted by Hughes *et al.* (2) for a cantilever beam and Eqs. (24a) and (24b) show that the reduced integration element leads to excellent results over a wide range of values of the beam slenderness γ. The numerical solution only deteriorates when the aspect ratio of the element h/t exceeds a critical value $(h/t)_c$ which depends on the computer word length. In the work of Hughes *et al.* (2) this critical aspect ratio was $10^4/16$. To eliminate this difficulty, these authors propose to multiply the shear stiffness by $(h/t)^2_c/(h/t)^2$ when $(h/t) > (h/t)_c$. This modification is equivalent to replacing γ_e in Eq. (23) by $\gamma_c = \sqrt{\kappa/(1+\nu)}\,(h/t)_c$ if $\gamma_e > \gamma_c$.

Prior to the work of Hughes *et al.* (2), Fried (1) had considered a 5-degree-of-freedom element characterized by the shape functions

$$w(h\xi) = (2\xi^2 - 3\xi + 1)\, w_1 + (2\xi^2 - \xi) w_2 + 4(\xi - \xi^2) w_3 \tag{26a}$$

$$\hat{\theta}(h\xi) = (1 - \xi)\, \hat{\theta}_1 + \xi\, \hat{\theta}_2 \tag{26b}$$

where w_1, $\hat{\theta}_1$, w_2 $\hat{\theta}_2$ are the end displacements and rotations and w_3 is the lateral displacement at the center of the element. The resulting bending and shear stiffness matrices obtained by *exact* integration of the strain energy are

$$[k_b'''] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{27}$$

$$[k_s'''] = \gamma_e^2 \begin{bmatrix} 14 & 5 & -16 & 2 & 1 \\ 5 & 2 & -4 & -1 & 1 \\ -16 & -4 & 32 & -16 & 4 \\ 2 & -1 & -16 & 14 & -5 \\ 1 & 1 & 4 & -5 & 2 \end{bmatrix}$$

for the five degrees-of-freedom w_1, $\hat{\theta}_1$, w_3, w_2 and $\hat{\theta}_2$. The interior degree-of-freedom w_3 can be eliminated by static condensation. From the 3rd row of Eqs. (27) and (28) it is found that the interior equilibrium condition is

$$w_3 = \frac{w_1 + w_2}{2} + \frac{\hat{\theta}_1 - \hat{\theta}_2}{8} \tag{29}$$

which, in the light of Eqs. (26a) and (26b), implies that $d(dw/dx - \theta)/dx = 0$. The interesting result is that once the internal degree-of-freedom w_3 is eleminated by use of Eq. (29) the resulting 4×4 condensed matrices $[k_b''']$ and $[k_s''']$ are indentical to the corresponding matrices $[k_b'']$ and $[k_s'']$ for the 4-degree-of-freedom reduced integration element considered by Hughes *et al.* (2). Thus, the 4-degree-of-freedom reduced integration element of Hughes *et al.* (2) is equivalent to the five-degree-of freedom full integration element of Fried (1).

Fried (1) found that for a sufficiently large element slenderness ratio the global stiffness matrix becomes ill-conditioned and numerical difficulties are encountered. In particular, for a rectangular cantilever beam of depth t, Fried found that the limiting elemento slenderness ratio h/t for the case $\kappa = 1$, $\nu = 0$ is

$$\frac{h}{t} = \frac{10^{(s-1)/2}}{N^{1.85}} \tag{30}$$

where s is the number of decimals in the particular computer used. To eliminate this problem, Fried (1) proposes to use a fictitious value for the thickness $t_B = h/c$, or, equivalently, a fictitious value for the element slenderness ratio $(\gamma_e)_B = c$ so that the discretization error and the shear correction are balanced. By numerical experimentation he found that the value $(\gamma_e)_B = c = \sqrt{2}$ leads, in the case of a cantilever beam, to the exact tip deflection for a Bernoulli beam. This result is much more general than the case considered by Fried (1). Indeed, substitution of $\gamma_e^2 = 2$ into Eq. (23) shows, that, for this value of γ_e the matrix $[k_e'']$ becomes identical to the 'standard' stiffness matrix for a Bernoulli beam and, consequently, the solution for the generalized nodal displacements will coincide with the exact solution for the Bernoulli beam for any combination of loads applied at the nodes. If γ^2 in Eq. (24a) is replaced by $\gamma_B^2 = 2N^2$, which corresponds to $(\gamma_e)_B = \sqrt{2}$, then the last two terms in Eq. (24a) cancel each other and the nodal results coincide with the exact solution for the Bernoulli beam. This result reveals that the success of the residual energy balancing of Fried (1, 7) stems from the fact that the discretization error and the shear correction act in opposite directions and that when these terms are balanced the exact shearless solution is obtained. Without the sign difference, balancing terms would only double the error.

Clearly, the corrections for the ill-conditioning problem proposed by Fried (1) and Hughes *et al.* (2) are equivalent to the use of a fictitious value for the element slenderness ratio γ_e. The corrections differ in that Fried (1) would use a value of $\gamma_e = \sqrt{2}$ while Hughes *et al.* (2) would use a much larger value $\gamma_e = \sqrt{\kappa/(1 + \nu)}(h/t)_e$.

The linear 4-degree-of-freedom C^0-element with reduced integration considered by Hughes *et al.* (2) has been also discussed by MacNeal (6) who found, by comparison of the strain energy for this element with that for a cubic displacement function, that the element behaviour can be improved by replacing the shear rigidity (κGA) by an effective value $(\kappa GA)_T$ defined by

$$\frac{1}{(\kappa GA)_T} = \frac{1}{\kappa GA} + \frac{h^2}{12EI} \tag{31}$$

where the second term on the right-hand-side is called the residual bending flexibility. This same result can be obtained in a slightly different way by comparison of the element stiffness matrix $[k_e'']$ given by Eq. (23) with the stiffness matrix $[k_e]$ for the 'standard' Timoshenko beam element given by Eq. (8). The two matrices will be equal if γ_e in Eq. (23) is replaced by $(\gamma_e)_T = \sqrt{(\kappa GA)_T\, h^2/6EI}$ in such a way that

$$(\gamma_e)_T^2 = \frac{\gamma_e^2}{1 + (\gamma_e^2/2)} = \frac{2}{1 + 2\gamma_e^{-2}} \tag{32a}$$

$$\frac{3}{2} + (\gamma_e)_T^{-2} = 2 + \gamma_e^{-2} \tag{32b}$$

$$\frac{3}{2} - (\gamma_e)_T^{-2} = 1 - \gamma_e^{-2} \tag{32c}$$

while the flexural rigidity. EI is left unchanged. It is easy to verify that Eqs. (32b) and (32c) are satisfied if $(\gamma_e)_l$ is defined by Eq. (32a) which is equivalent to Eq. (31).

Thus, if the shear rigidity is modified according to Eq. (31), or, equivalently, if the element slenderness ratio γ_e is replaced by $(\gamma_e)_T$ given by Eq. (32a) while the flexural rigidity EI is kept unchanged, then, the four-degree-of-freedom reduced-integration C^o-element leads to exactly the same total stiffness matrix as the 'standard' Timoshenko beam element. With this modification, the element gives exact nodal deflections and rotations for forces and moments acting at the nodes. When full-integration is used the same results can be obtained by modifying both the flexural and shear rigidities.

If $\gamma_e \rightarrow \infty$ then Eq. (32a) leads to $(\gamma_e)_T \rightarrow (\gamma_e)_B = \sqrt{2}$ which corresponds to the optimal modification proposed by Fried (1). Finally, Eq. (32a) implies that $((\gamma_T)^{-2} - 2N^2)^{-1} = (\gamma)^{-2}$ where $\gamma_T = N(\gamma_e)_1$. Substitution of γ by γ_T in Eq. (24a) and use of this relation confirms that the modified element leads to the exact nodal solution for a Timoshenko beam.

Conclusions

A simple analytical description of the problem of locking of a linear four-degree-of-freedom C^o Timoshenko beam element has been presented. It has been shown that the total stiffness matrix of the element is proportional to the stiffness matrix for the 'standard' non-locking Timoshenko beam element and that the factor of proportionality tends to infinity as the aspect ratio of the element tends to infinity. The nodal deflection and rotation of a beam discretized by a number of equal C^o-elements are then exactly proportional to the solution based on 'standard' elements. The factor of proportionality tends to zero as the aspect ratio of the element increases and hence locking results. This simple description of the problem of locking also suggests a new and simple way of avoiding the problem, namely, to scale the bending and shear rigidities by a common factor selected so that the factor of proportionality is equal to one. An impractical alternative would be to use elements with length shorter than the radius of gyration of the cross-section.

The traditional remedy for locking has been the use of selective reduced integration coupled with a scaling of the shear strain energy (2) or with the use of a modified shear rigidity (6). It has been shown that for the simple linear four-degree-of-freedom C^0-element considered by Hughes *et al.* (2), the combination of reduced integration and modification of the shear rigidity as proposed by MacNeal (6) leads to an element stiffness matrix which is identical to that for the nonlocking 'standard' Timoshenko beam element. The resulting nodal displacement and rotations are then exact for loads applied to the nodes. Reduced integration combined with reduction of the shear rigidity leads to the same result as full integration combined with reduction of both the bending and shear rigidities.

It has also been shown that the five-degree-of-freedom C^0-element considered by Fried (1) involving a quadratic displacement, a linear rotation of the cross-section and full integration, leads, after static condensation of the internal degree of freedom, to the same stiffness matrix as that for the four degree-of-freedom reduced integration C^0-element considered by Hughes *et al.* (2). The equilibrium condition at the internal node corresponds to a condition of constant shear strain along the element. The equivalence of these two elements gives some insight into the effect of reduced integration which eliminates the contribution of a linearly varying component of the shear strain. The equivalence also shows that the ill-conditioning studied by Fried (1) also affects the element considered by Hughes *et al.* (2). The remedies proposed by Fried (1), who uses a fictituous depth for the beam proportional to the length of the element, and Hughes *et al.* (2), who divided the shear strain energy by a factor proportional to the square of the element slenderness ratio, are similar in that they are equivalent to the use of a fictitious element slenderness ratio but differ on the value for this ratio. Fried (1) suggests the use of a relatively small value $(\gamma_e)_B = \sqrt{2}$ for which the element stiffness matrix becomes indentical to that for the 'standard' Bernoulli beam element. Hughes *et al.* (2), on the other hand, recommend a large value corresponding to the largest value of γ_e for which accurate numerical results are obtained. The modification of the shear rigidity proposed by MacNeal (6), which includes, as a limiting case, the correction proposed by Fried (2), appears to be the best solution to the ill-conditioning problem.

Analytical solutions for a cantilever Timoshenko beam subjected to an end load and discretized into N equal linear C^0-elements for both full and reduced integration have been obtained. These solutions give simple descriptions of the effects of locking and reduced integration and show that the energy balancing approach suggested by Fried (1) works as a result of the shear correction cancelling the error due to the discretization. Comparisons of the results from the analytical solutions with numerical results quoted extensively in the literature have uncovered a number of minor and major errors.

APPENDIX
ON THE PROPAGATION OF ERRORS

Initial comparisons of published numerical results with the exact solution obtained here [Eq. (17)] for the normalized response of a discretized cantilever Timoshenko beam proved unsuccessful. The search for the source of the discrepancies led to the discovery of a proliferation of minor and more serious errors as the initial numerical results of Hughes *et al.* (2) were reported in subsequent papers and texts.

The chain starts with the work of Hughes *et al.* (2) who considered the case of a uniform rectangular cantilever beam subjected to an end load and characterized by $E = 1000$, $G = 375$ ($\nu = 1/3$), depth $T = 1$, with $b = 1$, lenght $l = 4$ and $\kappa = 5/6$. For this 'deep' beam $\gamma^2 = [(5/6)(4/3)]^*(4)^2 \doteq 10$. To test the Bernoulli-Euler limit, Hughes *et al.* (2) also considered the same beam but with G increased to 375×10^5. This 'thin' beam is characterized by $\gamma^2 = 10^6$. The numerical results obtained by Hughes *et al.* (2) for the tip displacement normalized by the exact solution for the Timoshenko beam are compared in Table A. I with the corresponding

TABLE A.I.

COMPARISON OF NUMERICAL RESULTS OBTAINED BY HUGHES *et al.* (1977) FOR THE NORMALIZED TIP DISPLACEMENT OF CANTILEVER BEAMS WITH THE ANALYTICAL SOLUTIONS OF THE DISCRETIZED EQUATIONS FOR FULL AND REDUCED INTEGRATION

	Reduced Integration		Full Integracion			
	Hughes *et al.* (1977) and present study		Hughes *et al.* (1977)		Present Study	
N	$\gamma^2 = 10$	$\gamma^2 = 10^6$	$\gamma^2 = 10$	$\gamma^2 = 10^6$	$\gamma^2 = 10$	$\gamma^2 = 10^6$
1	0,762	0,750	0,0416*	$0{,}200 \times 10^{-4}$*	0,167	$0{,}200 \times 10^{-5}$
2	0,940	0,938	0,445	$0{,}800 \times 10^{-4}$*	0,444	$0{,}800 \times 10^{-5}$
4	0,985	0,984	0,762	$0{,}320 \times 10^{-3}$*	0,762	$0{,}320 \times 10^{-4}$
8	0,996	0,996	0,927	$0{,}128 \times 10^{-3}$	0,928	$0{,}128 \times 10^{-3}$
16	0,999	0,999	0,981	$0{,}512 \times 10^{-3}$	0,981	$0{,}512 \times 10^{-3}$

analytical results given by Eq. (17) for full integration and Eq (25) for reduced integration. The numerical results obtained by Hughes *et al.* (2) for the case of reduced integration coincide exactly with the results from Eq. (25). For full integration, some of the numerical results obtained by Hughes *et al.* (2) differ from those based on Eq. (17) and appear to be in error. In particular, the numerical results for $\gamma^2 = 10^6$ and N = 1, 2, 4, appear to suffer from a tipographical error in the exponents. Hand calculation for the case N = 1, $\gamma^2 = 10$ confirms that the result listed by Hughes *et al.* (2) is in error.

In a subsequent paper, Malkus and Hughes (4) use the same numerical results listed by Hughes *et al.* (2) but attribute these results to beams characterized buy l = 4, b = 1, κ = 1, t = 0.79 (deep t beam) and t = 2.5 x 10^{-3} (thin beam). The values listed for the depth t of the beams are in error. The correct values of the depth t required to match the values of $\gamma^2 = 10$ and $\gamma^2 = 10^6$ for the deep and thin beams are $t = 4/\sqrt{10} = 1.2649$ and $t = 4 \times 10^{-3}$, respectively.

Zienkiewicz (3) in his text (Table 11.1, p. 291) reproduces the results obtained by Hughes *et al.* (2) but introduces two additional missprints. The result for the deep beam, N = 1 and reduced integration is listed as 0.752 instead of the correct value of 0.762. Also the result for the thin beam, N = 2 and full integration is listed as 0.3×10^{-4}, while the result given by Hughes *et al.* (2) is 0.8×10^{-4} and the correct value is 0.8×10^{-5}. In addition, Zienkiewicz (3) introduces the non-dimensional parameter

$$\alpha' = \frac{\kappa G l^2}{E t^2} = \frac{\gamma^2}{2} \tag{A.1}$$

and atributes the results for the deep and thin beams to values of $\alpha' = 7.2 \times 10^5$, respectively. These value for α' are in error. The correct values for the deep and thin beams are $\alpha' = 5$ and $\alpha' = 5 \times 10^5$, respectively. The error appears to stem from the use of $\kappa = 6/5$ instead of the correct value $\kappa = 5/6$.

Hinton and Owen (5) in their text also use the results of Hughes *et al.* (2) to illustrate the 'locking' effects. In Fig. 5.9 (p. 145) they reproduce some of the results shown in Fig. 6.16 of Hughes *et al.* (2) and claim in the caption that the results demonstrate the locking effect for a Timoshenko beam modelled with 16 elements and *full* integration. In reality, the results correspond to a demonstration by Hughes *et al.* (2) of the ill-conditioning effects stemming from the vanishing of the bending stiffness as a result of the finite computer word length when *reduced* integration is used. In Fig. 5.10 (p. 146) (5) also reproduce results shown in Fig.

6.16 of Hughes *et al.* (2) and claim that it describes results without locking obtained for a Timoshenko beam modelled with 16 elements and reduced integration. In reality, the results correspond to a modification of the reduced integration approach proposed by Hughes *et al.* (2) to eliminate ill-conditioning when the element slenderness ratio becomes extremely large.

The normalized tip displacements presented in Fig. 6.16 of Hughes *et al.* (2) for a cantilever beam discretized into 16 equal 4-degree-of-freedom reduced integration elements are larger than 1.0 for values of l/t smaller than 20. This suggests that these results are normalized by the exact displacement for a Bernoulli beam rather than by the exact displacement for a Tismoshenko beam as done elsewhere in the paper. The numerical results shown also appear to imply that values of $\kappa = 1$ and $\nu = 0$ were used which are different from the values used elsewhere in the paper. The point in Fig. 6.16 of Hughes *et al* (2) corresponding to l/t = 4 appears to be missploted.

Finally, Fried (1) reports that the relative error (with respect to the result for a Bernoulli beam) of the tip deflection of a cantilever beam of length 1 modelled by equal elements of length h is given by 0,4 $(h/1)^2$. The beam is characterized by $\gamma = 0$, $\kappa = 1$, t = h, b = 1 and the five-degree-of-freedom C^0-elements obtained by full integration, which are equivalent to the four-degree-of-freedom reduced- integration elements considered by Hughes *et al.* (2), are used. In this case, $\gamma_e = 1$ and $\gamma = 1/h = N$. Eq. (29a) indicates that the relative error with respect to the result for the Bernoulli beam is, in this case, $[\gamma^{-2} - (2N^2)^{-1}]/2 = 0.25\ (h/1)^2$ which differs from the numerical result reported by Fried (1).

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