LOS MÉTODOS DE INTEGRACION

POR

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(Continuacion)

8. $\int_x \pi(m_0 x + c x^{s+1}) a^2 x^{2n} = a^2 \pi \int (m_0 x^{2n+1} + c x^{s+2n+1}) dx$

$$a^2 \pi \left(\frac{m_0}{2n+2} x^{2n+2} + \frac{c}{s+2n+2} x^{s+2n+2} \right)$$

(Perry, 76)

9. $\int_x (a+x)(b-x)\,dx = \int_x [ab-(a-b)x-x^2]$

$$= abx - \frac{1}{2}(a-b)x^2 - \frac{1}{3}x^3$$

10. $\int (ax^m + b)a^1x^n + b^1)dx = \int_x (aa^1x^{m+n} + a^1bx^n +$

$$ab^1x^m + bb^1)$$

$$= \frac{aa^1x^{m+n+1}}{m+n+1} + \frac{a^1bx^{n+1}}{n+1} + \frac{ab^1x^{m+1}}{m+1} + bb^1x.$$

11. $\int_x (ax+b)(cx^2+px+q) = \frac{1}{12}x[3acx^3+4(ap+bc)x^2$

$$+6(aq+bp)x+12bq]$$

12. $\int (x^2\dot{x} + a\,x\,\dot{x} + b\,b\,\dot{x}) = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + b^2x.$

(Mac Laurin, II, 599)

En la notacion anterior, debida a Newton, $\dot{x} = d\,x$.

49. *Desarrollo de una potencia.*—Para integrar la potencia de un polinomio, tal como

$$(p_0\,x^m + p_1\,x^{m-1} + p_2\,x^{m-2} + \ldots\ldots + p_m)^n,$$

se desarrolla conforme a las reglas del Aljebra i en seguida se integra cada uno de sus términos.

1. $\int_x (a+b\,x)^2 = \int_x (a^2 + 2\,a\,b\,x + b^2x^2)$

$$= \int_x a^2 + \int_x 2\,a\,b\,x + \int_x b^2x^2$$

$$= a^2x + a\,b\,x^2 + \frac{1}{3}\,b^2x^3.$$

2. $\int (a\,x^2 - b)^2 = \frac{1}{6}\,a^2x^6 - \frac{1}{2}\,a\,b\,x^4 + \frac{1}{2}\,b^2x^2$

(Timmermans, 247)

3. $\int_x (1 - a\,x^2)^4 = x - \frac{4}{3}\,a\,x^3 + \frac{6}{5}\,a^2x^5 - \frac{4}{7}\,a^3x^7$

$$+ \frac{1}{9}\,a^4x^9.$$

4. $\int (1+x)^n\,dx$. Desarrollemos por el binomio de Newton:

$$(1+x)^n = 1 + \frac{n}{1!}\,x + \frac{n\,(n-1)}{2!}\,x^2 + \frac{n\,(n-1)\,(n-2)}{3!}\,x^3$$

$$+\frac{n(n-1)\ldots(n-3)}{4!}x^4+\ldots+x^n.$$

Reemplazamos e integramos cada término:

$$\int(1+x)^n = x + \frac{n}{2!}x^2 + \frac{n(n-1)}{3!}x^3 + \frac{n(n-1)(n-2)}{4^1!}x^4$$

$$+\frac{n(n-1)\ldots(n-3)}{5!}x^5+\ldots+\frac{1}{n+1}x^{n+1}.$$

5. $\int_x(a^n+x^n)^m$. Se puede proceder así:

$$(a^n+x^n)^m = a^{mn}\left[1+\left(\frac{x}{a}\right)^n\right]^m.$$

$$= a^{nm}\left[1+\frac{m}{1!\,a^n}x^n+\frac{m(m-1)}{2!\,a^{2n}}x^{2n}+\ldots+\left(\frac{x}{a}\right)^{mn}\right]$$

$$\therefore \int_x(a^n+x^n)^m = a^{nm}\left[x+\frac{m}{1!\,a^n(n+1)}x^{n+1}\right.$$

$$+\frac{m(m-1)}{2!\,a^{2n}(2n+1)}x^{2n+1}+.$$

$$\left.+\frac{1}{a^{mn}(mn+1)}x^{mn+1}\right]$$

6. $\int_x(a+bx+cx^2)^2 = a^2x+abx^2+\frac{1}{3}(2ac+b^2)x^3$

$$+\frac{1}{2}bx^4+\frac{1}{5}c^2x^5.$$

(Bossut, I, 253)

7. $\int(a+bx+cx^2+\ldots)^n dx$

$$=a^n\int dx$$

$$+\frac{n}{1!}a^{n-1}\int(bx+cx^2+\ldots)$$

$$+\frac{n(n-1)}{2!}a^{n-2}\int(bx+cx^2+\ldots.)^2$$

$$+\frac{n(n-1)(n-2)}{3!}\int(bx+cx^2+\ldots)^3$$

$$+\ldots\ldots$$

8. $\int x^2(a-x)^3dx$

$$=\int(a^3x^2-3a^2x^3+3ax^4-x^5)dx$$

$$=\frac{1}{2}a^3x^3-\frac{3}{4}a^2x^4+\frac{3}{5}ax^5-\frac{1}{6}x^6.$$

9. $\int(a+bx^2)^3(a-bx^3)^2dx$

$$=a^5x-a^4bx^3-\frac{1}{2}a^4bx^4+\frac{3}{5}a^3b^2x^5-a^3b^2x^6$$

$$+\frac{1}{7}(a^2b^3+a^3b^2)x^7-\frac{3}{4}a^2b^3x^8+\frac{1}{3}a^2b^3x^9$$

$$-\frac{1}{5}ab^4x^{10}+\frac{3}{11}ab^4x^{11}+b^5x^{13}.$$

10. $\int\frac{a(x-b)^3}{nx^4}dx=\frac{a}{n}\int(x-b)^3x^{-4}dx$

$$= \frac{a}{n} \left(L\,x + \frac{3\,b}{x} = \frac{3\,b^2}{2\,x^2} + \frac{b^3}{2\,x^3} \right)$$

(Pauly, 163)

11. $\int_x (a\,x^n + b)^m\, x^r$

$$= \frac{a^m\, x^{m\,n+r+1}}{m\,n+r+1}$$

$$+ \frac{m}{1!} \cdot \frac{a^{m-1}\,b}{n\,(m-1)+r+1}\, x^{n\,(m-1)+r+1}$$

$$+ \frac{m(m-1)}{2!} \cdot \frac{a^{m-2}\,b^2}{n\,(m-2)+r+1}\, x^{n\,(m-2)+r+1}$$

$+ \ldots\ldots$

50. *Simplificaciones.*—Las fracciones, ántes de integrarlas, deben ser reducidas á su mas simple espresion, para lo cual se descomponen en factores sus dos términos.

1. $\int_x \frac{x^2-1}{x-1} = \int_x \frac{(x+1)\,(x-1)}{x-1} = \int (x+1) = \frac{1}{2}x^2 + x$

2. $\int_x \frac{1+x^3}{1+x} = \int_x (1 - x + x^2) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

3. $\int_x \frac{x^2-3\,x+2}{x-1} = \int_x \frac{(x-1)\,(x+2)}{x-1} = \int_x (x+2)$

$$= \frac{1}{2}\,x^2 + 2\,x$$

4. $$\int_x \frac{a^4 - a^4}{x^2 + a^2} = \int_x (x^2 + a^2) = \frac{1}{3} x^3 + a^2 x$$

5. $$\int_x \frac{x^5 - 1}{x - 1} = \int_x (x^4 + x^3 + x^2 + x + 1)$$

$$= \frac{1}{5} x^5 + \frac{1}{4} x^4 + \frac{1}{3} x^3 \frac{1}{2} x^2 + x.$$

6. $$\int_x \frac{x^6 + a^6}{x^2 + a^2} = \int_x (x^4 - a^2 x^2 + a^4)$$

$$= \frac{1}{5} x^5 + \frac{a^2}{3} x^2 + a^4 x$$

7. $$\int_x \frac{6x^2 - 8ax}{9ax - 12a^2} = \int \frac{2x(3x - 4a)}{4a(3x - 4a)} = \frac{2}{3a} \int x = \frac{1}{3a} x^2$$

8. $$\int_x \frac{20(x^3 - a^3)}{(5x^2 + 2ax) + a(3x + 5a)} = \int_x 4(x - a)$$

$$= 4\left(\frac{1}{2} x^2 - ax\right).$$

9. $$\int_x \frac{x^4 + x^3 + x^2 + x + 1}{x^5 - 1} = \int_x \frac{1}{x - 1} = L(x - 1)$$

51. *Descomponer por division.*—Para integrar una fraccion cuyo numerador es de un grado superior o igual al del denominador, se efectúa la division.

1. $$\int_x \frac{x + 1}{x - 1} = \int_x \left(1 + \frac{2}{x - 1}\right) = x + 2L(x - 1)$$

2. $\int_x \frac{x}{a+bx} = \int_x \left(\frac{1}{b} - \frac{a}{b(bx+a)} \right) = \frac{x}{b} - \frac{a}{b^2} L(a+bx)$

(Peacock, 269)

3. $\int_x \frac{x^2}{x-1} = \int_x \left(x + 1 + \frac{1}{x-1}\right) = \frac{1}{2}x^2 + x + L(x-1)$

4. $\int_x \frac{x^2+1}{x+1} = \int_x \left(x - 1 + \frac{2}{x+1}\right) = \frac{1}{2}x^2 - x + 2L(x+1)$

5. $\int_x \frac{x^2-x+2}{x+1} = \int_x \left(x - 2 + \frac{4}{x+1}\right)$

$$= \frac{1}{2}x^2 - 2x + 4\,L\,(x+1)$$

6. $\int_x \frac{x^2-2x+5}{x-1} = \int_x \left(x^2 + x - 1 + \frac{4}{x-1}\right)$

$$= \frac{1}{3}x_3 + \frac{1}{2}x^2 - x + L\,(x-1)$$

(De Comberousse, IV, 684)

7. $\int_x \frac{x^2+x-2}{x^2+2x-3} = \int_x \frac{x-2}{x-3} \int_x \left(1 - \frac{5}{x+3}\right)$

$$= x + L\,(x+3)^5.$$

8. $\int_x \frac{x^3}{a+bx} = \int_x \left[\frac{x^2}{b} - \frac{ax}{b^2} + \frac{a^2}{b^3} - \frac{a^3}{b^3(bx+a)} \right]$

$$= \frac{x^3}{3b} - \frac{ax^2}{2b^2} + \frac{a^2x}{b^3} - \frac{a^3}{b^4}\,L\,(bx+a)$$

9. $$\int_x \frac{2-x^3}{1-x} = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x - L(1-x)$$

(Bertrand, II, 131)

10. $$\int_x \frac{x^4}{a+bx^2} = \int_x \left[\frac{x^2}{b} - \frac{a}{b^2} + \frac{a^3}{b^2(bx^2+a)}\right]$$

$$= \frac{x^3}{3b} - \frac{ax}{b^2} + \frac{a^3}{b^2} \cdot \frac{1}{\sqrt{ab}} \text{ arc tg } \sqrt{\frac{b}{a}}x.$$

11. $$\int_x \frac{x^5-1}{x+1} = \frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3} + \frac{1}{2}x^2 + x$$

12. $$\int_x \frac{\operatorname{sen}^2 x - \operatorname{sen} x - \cos x}{\operatorname{sen} x - 1} = -\cos x + L(\operatorname{sen} x - 1)$$

13. $$\int_x \frac{a^{x+1} - a^{2x-1} - xa^x + xa^2x - 1}{a-x} \int_x = \left(a^x - a^2x - \frac{1}{a-x}\right)$$

$$= \frac{x^x}{La} - \frac{1}{2}a^2x^2 + L(a-x)$$

14. $$\int_y \frac{y^n}{y-a} = \frac{y^n}{n} + \frac{a\,y_{n-1}}{n-1} + \frac{a\,y^{n-1}}{n-2} + \ldots + a^{n-1}y$$

$$+ \frac{a^n}{n} L(y-a).$$ (Mac Laurin, II, 629)

15. $$\int_x \frac{\left(\frac{x}{a}\right)^3 + \left(\frac{a}{x}\right)^3 - 3\left[\left(\frac{x}{a}\right)^2 - \left(\frac{a}{x}\right)^2\right] + 4\left(\frac{x}{a} + \frac{a}{x}\right)}{\frac{x}{a} + \frac{a}{x}}$$

$$= \frac{x^3}{3a^2} - \frac{3x^2}{2a} - 3aLx + \frac{a^2}{x} + 3x$$

(R. Potts, V, 17)

52. *Descomposicion de las fracciones.*—La fraccion

$$\frac{F(x)}{f(x)} = \frac{a x^m + b x^{m-1} + \dots + p}{f(x)}$$

se descompone en las fracciones mas sencillas

$$\frac{F(x)}{f(x)} = a \frac{x^m}{f(x)} + b \frac{x^{m-1}}{f(x)} + \dots + \frac{p}{f(x)}$$

por simple division o segun los métodos que a continuacion se indican.

1. $\int_x \frac{a+x^2}{x} = \int_x \left(\frac{a}{x} + x\right) = aLx + \frac{1}{2}x^2$

2. $\int_x \frac{x+1}{x^2+1} = \int_x \left(\frac{x}{x^2+1} + \frac{1}{x^2+1}\right) = \frac{1}{2}L(x^2+1)$

$$+ \operatorname{arc\,tg} x$$

(Axel Karnack, 190)

3. $\int_x \frac{ax}{b+cx} = \int_x \frac{acx}{c(b+cx)} = \int_x \frac{acx+ab-ab}{c(b+cx)}$

$$= \int_x \left(\frac{a}{c} - \frac{ab}{c(b+cx)}\right) = \frac{a}{c}x - \frac{ab}{c^2}L(b+cx)$$

(R. H. Smith, 61)

4. $\int_x \frac{(1+x^2)^2}{x} = \int_x \left(\frac{1}{x} + 12x^2 + x^3\right) = Lx$

$$+ \frac{2}{3}x^3 + \frac{1}{4}x^4.$$

5. $\int_x \frac{(1-x^2)^2}{x} = Lx - x^2 + \frac{1}{4}x^4.$ (Williamson, 14)

6. $\int_y \frac{(y-a)^3}{y^m} = \int_y (y^{3-m} - 3ay^{2-m} + 3a^2 y^{1-m} - a^3 y^{-m})$

$$= \frac{y^{4-m}}{4-m} - \frac{3ay^{3-m}}{3-m} + \frac{3a^2 y^{2-m}}{2-m} - \frac{a^3 y^{1-m}}{1-m}$$

(Perry, 281)

7. $\int_x \frac{a-bx}{a+x} = a L(c+x) - bx + c L(c+x)$

(Véase ejercicio 3)

8. $\int_x \frac{1-x}{1+x^2} = \text{art tg } x - \frac{1}{2} L(1+x^2)$

9. $\int_x \frac{1}{1-x^2}$. Esta integral se considera como fundamental. La descomponemos como se indica:

$$\frac{1}{1-x^2} = \frac{1 + \frac{1}{2}x - \frac{1}{2}x}{1-x^2} = \frac{\left(\frac{1}{2} + \frac{1}{2}x\right) + \left(\frac{1}{2} - \frac{1}{2}x\right)}{(1+x)(1-x)}$$

$$= \frac{\frac{1}{2}(1+x) + \frac{1}{2}(1-x)}{(1+x)(1-x)} = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right)$$

$$\therefore \int_x \frac{1}{1-x^2} = \frac{1}{2}\int_x \left(\frac{1}{1+x} + \frac{1}{1-x}\right)$$

$$= \frac{1}{2} L(1+x) - \frac{1}{2} L(1-x)$$

$$= L\sqrt{\frac{1+x}{1-x}}$$

10. $$\int_x \frac{1}{x^2-1} = L\sqrt{\frac{1-x}{1+x}}$$

11. $$\int_x \frac{1}{a^2-x^2} = \frac{1}{a} L\sqrt{\frac{a+x}{a-x}}$$

12. $$\int \frac{dx}{a^2-b^2x^2} = \frac{1}{2a}\int \frac{(a-bx)+(a+bx)}{(a+bx)+(a-bx)}$$

$$= \frac{1}{ab} L\sqrt{\frac{a+bx}{a-bx}} \quad \text{(De Comberousse, 685)}$$

13. $$\int_x \frac{ax+b}{x^2+c^2} = \int_x \frac{ax}{x^2+c^2} + \int_x \frac{b}{x^2+c^2}$$

$$= 4 L\sqrt{a^2+c^2} + \frac{b}{c}\,\text{arc tg}\,\frac{x}{c}. \quad \text{(Hoüel, I, 213)}$$

14. $$\int_x \frac{A \mp Bi}{x-a \mp bi} = \int_x \frac{(a \mp Bi)(x-a \pm bi)}{(x-a)^2+b^2}$$

$$= \frac{1}{2} (A \mp B i) L [(x-a)^2 + b^2] + (B \pm a i) \text{ arc tg } \frac{x-a}{b}$$

(Moigno, II, 17)

15. $$\int_x \frac{dx}{x^3-1} = \int_x \frac{1}{3} \left(\frac{1}{x-1} - \frac{x+2}{x^2+x+1} \right)$$

$$= \frac{1}{6} L \left[\frac{(x-1)^2}{x^2+x+1} \right] - \frac{1}{\sqrt{3}} \text{ arctg } \frac{2x+1}{\sqrt{3}}$$

16. $$\int_x \frac{l+mx}{a+bx^2} = l \int_x \frac{1}{a+bx^2} + m \int_x \frac{x}{a+bx^2}$$

$$= \frac{l}{\sqrt{ab}} \text{ arc tg } \sqrt{\frac{b}{a}}\, x + \frac{m}{2b} L (a+bx^2)$$

(Roberts, 13)

17. $$\int_x \frac{ax^2+bx+c}{px+q} = \int_x \left(\frac{ax2}{px+q} + \frac{bx}{px+q} + \frac{c}{px+q} \right)$$

La integral se obtiene efectuando la division; las otras dos son conocidas:

$$y = a \left[\frac{x^2}{2p} - \frac{q}{p^2} x + \frac{q^2}{p^3} L (px+q) \right]$$

$$= b \left[\frac{1}{p} x - \frac{q}{p^2} L (px+q) \right] + \frac{c}{p} L (px+q).$$

18. $$\int_x \frac{Ax+B}{a+bx+cx^2} = \int_x \frac{ax + \frac{Ab}{2c} + B - \frac{Ab}{2c}}{a+bx+cx^2}$$

$$= \frac{A}{2c} \int_x \frac{2cx+b}{a+bx+cx^2} + \left(B - \frac{Ab}{2c}\right) \int_x \frac{1}{a+bx+cx^2}$$

$$= \frac{A}{2c} L(a+bx+cx^2) + \left(B - \frac{Ab}{2c}\right) \frac{2}{\sqrt{4ac-b^2}} \operatorname{arctg} \frac{2cx+b}{\sqrt{4ac-b^2}}$$

(Todhunter, 14)

19. $$\int_x \frac{ax+b}{x^2+px+q} = \int_x \left(\frac{a}{2} \frac{2x+p-p}{x^2+px+q} + b \frac{1}{x^2+px+q} \right)$$

$$= \left(b - \frac{ap}{2}\right) \int_x \frac{1}{x^2+px+q} + \frac{a}{2} \int_x \frac{2x+p}{x^2+px+q}$$

$$= \left(b - \frac{ap}{2}\right) \frac{2}{\sqrt{4q-p2}} \text{ art tg } \frac{2x+p}{\sqrt{4q-p2}}$$

$$+ \frac{a}{2} L(x^2+px+q).$$ (Gregory, 251)

20. $$\int_x \frac{3x^4-6x^2+1}{x^2-2x+2} = \int_x \left(3x^2+6x + \frac{1-12x}{x^2-2x+2} \right)$$

$$= x^3+3x^2-6 L(x^2-2x+2)+11 \text{ arc tg } (x-1).$$

(Timmermans, 259)

21. $\int \frac{Ax+B}{(x-\alpha)^2+\beta^2} dx$. Agreguemos $\alpha - \alpha = o$ al numerador:

$$\frac{Ax+B}{(x-\alpha)^2+\beta^2} = A \frac{x+\alpha-\alpha}{(x-\alpha)^2+\beta^2} + \frac{B}{(x-\alpha)^2+\beta^2}$$

$$= \frac{A}{2} \cdot \frac{2(x-\alpha)}{(x-\alpha)^2+\beta^2} + \frac{B+A\alpha}{(x-\alpha)^2+\beta^2}$$

$$\therefore y = \frac{1}{2} A \, L\left[(x-\alpha)^2+\beta^2\right] + \frac{B+A\alpha}{\beta} \text{ arc tg } \frac{x-\alpha}{\beta}$$

(Briot, II, 503)

22. $$\int_x \frac{x+1}{\sqrt{1-x^2}} = \int_x \frac{x}{\sqrt{1-x^2}} + \int_x \frac{1}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2} + \text{arc sen } x.$$

(Moigno, II, 15)

23. $$\int \frac{x\,dx}{(x^2+p\,x+q)^{\frac{3}{2}}} = \frac{1}{2}\int \frac{2x+p}{(x^2+p\,x+q)^{\frac{3}{2}}}dx$$

$$-\frac{p}{2}\int \frac{dx}{(x^2+p\,x+q)^{\frac{3}{2}}}$$

$$= (x^2+p\,x+q)^{-\frac{1}{2}} - \frac{2}{4q-p^2} \cdot \frac{p\,x+2q}{\sqrt{x^2+x+q}}$$

(Brahy, 24)

24. $\int e^x \cdot \frac{x^2+1}{(x+1)^2}\,dx$. La fraccion se descompone así:

$$\frac{x^2+1}{(x+1)^2} = \frac{x^2+1-1+1}{(x+1)^2} = \frac{x^2+1}{(x+1)^2} + \frac{2}{(x+1)^2}$$

$$= \frac{x-1}{x+1} + \frac{2}{(x-1)^2}.$$

Pero $d\,\dfrac{x-1}{x+1} = \dfrac{2}{(x-1)^2}$

$$\therefore y = \int e^x \left(\frac{x-1}{x+1}\, dx + d\,\frac{x-1}{x+1} \right)$$

$$= \int \left(\frac{x-1}{x+1}\, de^x + e^x\, d\,\frac{x-1}{x+1} \right)$$

$$= \int (u\,dv + d\,u) = \int d\,(uv)$$

$$= e^x .\ \frac{x-1}{x+1} \quad \text{(Grégory, 253)}$$

25. $$\int_x \frac{x . e^x}{(1+x)^2} = \int_x e^x \left[\frac{x+1-1}{(1+x)^2} \right]$$

$$= \int_x e^x \left(\frac{dx}{1+x} + d\,\frac{1}{1+x} \right)$$

$$= \int_x \left(\frac{de^x}{1+x} + e^x\, d\,\frac{1}{1+x} \right)$$

$$= e^x\,\frac{1}{1+x} .$$

26. $$\int_x e^x \frac{2-x^2}{\sqrt{1+x}\,\sqrt{(1-x)^3}} = \int_x e^x \left[\frac{1+(1+x)(1-x)}{\sqrt{(1+x)(1-x)(1-x)^2}} \right]$$

$$= \int_x \left(d\,e^x \sqrt{\frac{1+x}{1-x}} + e^x\, \sqrt{\frac{1+x}{1-x}} \right) = e^x \sqrt{\frac{1+x}{1-x}} \Big)$$

27. $$\int \frac{dx}{x\,L\,x} = \int_x \frac{1 + L\,x - L\,x}{x\,L\,x} \int_x \frac{1 + L\,x}{x\,L\,x} \int \frac{dx}{x}$$

$$= L\,(x\,L\,x) - L\,x = L\,(L\,x)$$

(Perry, 284)

28. $\int \frac{1 - x \operatorname{sen} \alpha}{1 - 2x \operatorname{sen} \alpha + x^2}\, dx$. Hacemos $1 = \operatorname{sen}^2 \alpha + \cos^2 \alpha$:

$$y = \int \frac{\cos^2 \alpha - \operatorname{sen} \alpha\,(x - \operatorname{sen} \alpha)}{1 - 2x \operatorname{sen} \alpha + x^2}\, dx$$

$$= \cos \alpha .\ \operatorname{arc\,tg} \frac{x - \operatorname{sen} \alpha}{\cos \alpha} - \operatorname{sen} \alpha\, L\, \sqrt{1 - 2x \operatorname{sen} x + x^2}$$

(Braby, 12)

53. *Fracciones parciales.*—Son las que resultan de la descomposicion de una fraccion comun en otras fracciones, cuya suma es igual a la primera.

1. $\int \frac{1}{1 - x^2}\, dx$ Integral importante i conocida.

Para descomponerla en otras dos, descomponemos el denominador en sus factores:

$$1 - x^2 = (1 + x)\,(1 - x);$$

en seguida nos valemos de las *indeterminadas* A, B, de modo que verifiquen la igualdad:

$$\frac{1}{(1 + x)\,(1 - x)} = \frac{A}{1 + x} + \frac{B}{1 - x};$$

hacemos desaparecer los denominadores, i se obtiene:

$$1 = A\ (1-x) + B\ (1+x). \qquad (1)$$

Como x es una variable, podemos suponer $x=1$ i resulta

$B = \frac{1}{2}$; i si suponemos $x = -1$, resulta $A = \frac{1}{2}$ luego,

$$\frac{1}{(1+x)\ (1-x)} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}.$$

$$\int_x \frac{1}{1-x^2} = \frac{1}{2}\int \frac{1}{1+x} + \frac{1}{2}\int \frac{1}{1-x}$$

$$= \frac{1}{2}\ L\ (1+x) - \frac{1}{2}\ L\ (1-x)$$

$$= L\sqrt{\frac{1+x}{1-x}}.$$

Método de los coeficientes indeterminados.—La ecuacion (1) se escribe así:

$$1 = (A+B) + (B-A)\ x;$$

o bien,

$$x^0 + 0\ .\ x^1 = (A+B)x^0 + (B-A)x^1$$

En Aljebra superior se demuestra que esta relacion es cierta cuando son iguales los coeficientes de las mismas potencias de x:

$$\left.\begin{array}{l} A+B=1 \\ B \quad A=0 \end{array}\right| \therefore \; B=\frac{1}{2}, \quad A=\frac{1}{2}.$$

2. $$\int \frac{1}{x^2-1}\, dx = \int \left(\frac{A}{x+1}+\frac{B}{x=1}\right) dx = L\sqrt{\frac{x-1}{x+1}}$$

3. $$\int \frac{1}{a^2-x^2}\, dx = \int \left(\frac{A}{a+x}+\frac{B}{a-x}\right) dx = \frac{1}{2a}\, L\, \frac{a+1}{a-x}$$

o bien se reduce a la forma elemental:

$$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\int \frac{d\,\frac{x}{a}}{1-\left(\frac{x}{a}\right)^2} = \frac{1}{a}\, L\sqrt{\frac{a+x}{a-x}}\,.$$

4. $$\int \frac{1}{x^2-a^2}\, dx = \frac{1}{a}\int \frac{d\,\frac{x}{a}}{\left(\frac{x}{a}\right)^2-1} = \frac{1}{a}\sqrt{\frac{x-a}{x+a}}$$

5. $$\int \frac{a}{a^2x^2-b^2}\, dx = \int \frac{1}{2b}\, L\, \frac{ax-b}{ax+b}.$$

6. $$\int \frac{1}{(x-a)(x-b)}\, dx = \int \left(\frac{A}{x-a}+\frac{B}{x-b}\right) dx;\; A=\frac{1}{a-b}$$

$$B=\frac{1}{b-a} \quad \therefore \quad y=\frac{1}{a-b}\, L\, \frac{x-a}{x-b}.$$

(Williamson, II, 8)

7. $$\int \frac{dx}{a+2bx+cx^2} = \int \frac{dz}{z^2+ac-b^2}. \quad (z=cx+b)$$

Si $a\,c - b^2$ es positivo,

$$y = \frac{1}{\sqrt{a\,c - b^2}} \text{ arc tg } \frac{c\,x + b}{\sqrt{a\,c - b^2}};$$

i si es negativo:

$$y = \frac{1}{2\sqrt{b^2 - a\,c}}\, L\, \frac{c\,x + b - \sqrt{b^2 - a\,c}}{c\,x + b + \sqrt{b^2 - a\,c}}$$

8. $\int \frac{d\,x}{x^2 + 9\,x + 20}$. Descompongamos el trinomio en sus factores:

$$x^2 + 9\,x + 20 = (x + 4)\,(x + 5);$$

luego,

$$\frac{1}{x^2 + 9\,x + 20} = \frac{A}{x + 4} + \frac{A}{x + 5}$$

$$\therefore\ 1 = A\,(x + 5) + B\,(x + 4)$$

Sea

$$x = -4 \ \therefore\ A = 1;\ x = -5 \ \therefore\ B = -1$$

$$\therefore\ \int \frac{d\,x}{x^2 + 9\,x + 20} = L\,\frac{x + 4}{x + 5}$$

9. $\frac{a\,x - b}{x^2 - c^2} = \frac{A}{x + c} + \frac{B}{x - c}$

$$\therefore\ a\,x - b = A\,(x - c) + b\,(x + c)$$

Sea

$$x = c \ \therefore\ a\,c - b = 2\,B\,c \ \therefore\ B = \frac{a\,c - b}{2\,c};$$

Sea

$$x = -c \ \therefore\ -a\,c - b = -2\,A\,c \ \therefore\ A = \frac{a\,c + b}{2\,c}:$$

$$\therefore \int \frac{a\,x - b}{x^2 - c^2}\,d\,x = \frac{a\,c + b}{2\,c}\,L\,(x + c) + \frac{a\,c - b}{2\,c}\,L\,(x - c)$$

(Timmermans, 261)

10. $$\frac{4\,x - 1}{3\,x^2 - 15\,x + 8} = \frac{1}{3}\left(\frac{A}{x - 2} + \frac{B}{x - 3}\right)$$

$$\therefore\ 4\,x - 1 = A\,(x - 3) + B\,(x - 2).$$

Sea

$$x = 2,\quad 8 + 1 = A \ \therefore\ A = 9;$$

Sea

$$x = 3,\ 12 + 1 = B \ \therefore\ B = 13:$$

$$\therefore\ \frac{4\,x - 1}{3\,x^2 - 15\,x + 18}\,d\,x = L\,\frac{(x - 3)^{\frac{13}{3}}}{(x - 2)^3}$$

(Sonnet, 207)

11. $$\int \frac{(2-4\,x)\,d\,x}{x^2-x-2} = \int \frac{2\,d\,x}{2-x} - \int \frac{2\,d\,x}{x+1}$$

$$= L\,\frac{1}{(x^2-x-2)^2}$$

(Francoeur, 360)

12. $$\frac{4\,x+1}{3\,(x-2)^2} = \frac{1}{3}\left[\frac{A}{x-2} + \frac{B}{(x-2)^2}\right]$$

$$\therefore\ 4\,x+1 = A\,(x-2)+B = A\,x-2\,A+B$$

$$\therefore\ A=4,\ B-2\,A=1\ \therefore\ B=9$$

$$\therefore\ y = \frac{4}{3}\,L\,(x-2) - \frac{1}{x-2}$$

13. $\frac{m\,x+n}{a\,x^2+b\,x+c}$. Sean α i β las raices *reales* i diferentes del trinomio:

$$\frac{m\,x+n}{(x-\alpha)\,(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$$

$$\therefore\ m\,x+n = A\,(x-\beta)+B\,(x-\alpha).$$

Sea

$x=\alpha$, queda: $m\,\alpha+n = A\,(\alpha-\beta)$

$$\therefore\ A = \frac{m\,\alpha+n}{\alpha-\beta}$$

Sea

$$x=\beta, \text{ queda: } m\,\beta+n = B\,(\beta-\alpha)$$

$$\therefore\ B = \frac{m\,\beta+n}{\alpha-\beta}$$

$$y = \frac{m\,\alpha+n}{\alpha-\beta}\,L\,(x-\alpha) + \frac{m\,\beta+n}{\alpha-\beta}\,L\,(x-\beta).$$

(Pauly, 165)

14. $\frac{2\,x-3}{x^2+2\,a\,x+3\,a^2}$. Agreguemos $2\,a-2\,a=0$; la fraccion se descompone así:

$$\frac{2\,x+2\,a}{x^2+2\,a\,x+3\,a^2} - \frac{2\,a+3}{x^2+2\,a\,x+3\,a^2}.$$

El trinomio se descompone en

$$2\,a^2+(x+a)^2.$$

La integral es,

$$y=L\,(x^2+2\,a\,x+3\,a^2)-\frac{2\,a+3}{a\,\sqrt{2}}\,\text{arc tg}\,\frac{x+a}{a\,\sqrt{2}}$$

(Brahy, 9)

15. $\frac{5\,x^3+1}{x^2-3\,x+2}$. Efectuamos la division i se obtiene el cuociente:

$$5x+15+\frac{35x-29}{x^2-3x+2}$$

Descompongamos la nueva fraccion:

$$35x-29=A(x-2)+B(x-1).$$

$$\therefore\ A=-6,\ B=41$$

$$\therefore\ y=\int\left(5x+15-\frac{6}{x-1}+\frac{41}{x-2}\right)dx$$

$$=\frac{5}{2}x^2+15x-6L(x-1)+41L(x-2)$$

(Todhunter, II, 31)

16. $$\frac{1}{x^3-x^2-x+1}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}.$$

$$\therefore\ 1=A(x^2-1)+B(x+1)+C(x-1)^2.$$

sea $x=1$, $B.(1+1)=1\ \therefore\ B=\frac{1}{2}$

sea $x=-1$, $C(-1-1)^2=1\ \therefore\ C=\frac{1}{4}$

Diferenciemos ahora:

$$0=2Ax+B+2C(x-1)$$

sea $x=1\ \therefore\ 0=2A+B=2A+\frac{1}{2}\ \therefore\ A=-\frac{1}{4}$

$$\therefore y = L\left(\frac{x+1}{x-1}\right)^{\frac{1}{4}} - \frac{1}{2}\,\frac{1}{x-1}$$

(Grégory, 262)

17. $$\frac{x}{x^3-1} = \frac{A\,x+B}{x^2+x+1} + \frac{C}{x-1}$$

$$\therefore x = (A+C)\,x^2 + (B+C-A)\,x + C - B:$$

$$A+C=0,\ B+C-A=1,\ C-B=0$$

$$\therefore \int \frac{x}{x^3-1}\,dx = \frac{1}{3}\left[L\,(x-1) - L\,\sqrt{x^2+x+1}\right.$$

$$\left. + \sqrt{3}\,\operatorname{arc}\left(\operatorname{tg} \frac{2\,x+1}{\sqrt{3}}\right)\right]$$

(Francoeur, 361)

18. $$\frac{du}{dx} = \frac{x-3}{x^3+1} = \frac{x-3}{(x+1)\,(x^2-x+x)}$$

Sea

$$\frac{x-3}{x^3+1} = \frac{A}{x+1} + \frac{M\,x+N}{x^2-x+1}$$

$$\therefore x-3 = A\,(x^2-x+1) + (x+1)\,(M\,x+N).$$

Sea

$x = -1$, se obtiene $A = -\frac{4}{3}$, valor que sustituimos arriba:

$$x-3 = -\frac{4}{3}\,(x^2-x+1) + (x+1)\,(M\,x+N)$$

$$\therefore\ M x+N = \frac{4x-5}{2}$$

$$\therefore \int_x \frac{x-3}{x^3+1} = -\frac{4}{3}\int_x \frac{1}{x+1} + \frac{1}{3}\int_x \frac{4x-5}{x^2-x+1}$$

La primera integral es: $-\frac{4}{3}$ L $(x+1)$; i la segunda se trasforma así:

$$\int \frac{4x-5}{\left(\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right)} = \int \frac{4x}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} - \int \frac{5}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= 2\,L\left[\left(x-\frac{1}{2}\right)^2+\frac{3}{4}\right] - 2\sqrt{3}\ \text{arc tg}\ \frac{2x-1}{\sqrt{3}}.$$

(Hall, 228)

19. $\frac{2x+3}{x^3+x^2-2x} = \frac{5}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{1}{x+2} - \frac{3}{2}\cdot\frac{1}{x}$

$$\therefore y = L\,\frac{(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}}\,(x+2)^{\frac{1}{6}}}$$

20. $\frac{du}{dx} = \frac{x^2-7x+1}{x^3-6x^2+11x+6}$. Hacemos sucesivamente $x=1, 2, 3$ i el denominador se reduce cada vez a cero· luego, la fraccion se descompone así:

$$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3};$$

$$\therefore\ x^2-7x+1=A(x-2)(x-3)+B(x-1)(x-3)$$
$$+C(x-1)(x-2)$$

Sea $x=1, 2, 3$; se obtiene sucesivamente:

$$A=-\frac{5}{2},\quad B=9,\ C=-\frac{11}{2}$$

$$\therefore\ u=L\,\frac{(x-2)^9}{\sqrt{(x-1)^5(x-3)^{11}}}.$$

21. $\int\frac{x^2-3x+5}{x^3+x^2-4x-4}\,dx$. Los factores del denominador son $x+1$, $x-2$, $x+2$; i las fracciones parciales:

$$\frac{A}{x+1}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$\therefore\ x^2-3x+5=A\left|\begin{array}{l}x^2\\ \\ +4\end{array}\right.+\ B\left|\begin{array}{l}x^2\\ +3x\\ +2\end{array}\right.+\ C\left|\begin{array}{l}x^2\\ -x\\ -2\end{array}\right.$$

Igualemos los coeficientes de las potencias iguales de x:

$$\left.\begin{array}{ll}A+B+C & =1\\ 3B-C & =-3\\ -4A+2B-2C & =5\end{array}\right|\ \therefore\ A=\frac{\begin{vmatrix}1&1&1\\-3&3&-1\\5&2&-2\end{vmatrix}}{\begin{vmatrix}1&1&1\\0&3&-1\\-4&2&-2\end{vmatrix}}=3,$$

$$B=\frac{1}{4},\ C=\frac{15}{4}$$

$$\therefore = -3\,L\,(x+1)+\frac{1}{4}\,L(x-2)+\frac{15}{4}\,L\,(x+2)$$

(Pauly, 168)

22. $$\frac{3x^2-2x+1}{x^4+2x^3-5x^2-6x}=\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-2}+\frac{D}{x+3}.$$

$$\therefore 3x^2+2x+1=A\left|\begin{array}{l}x^3\\+2x^2\\-5x\\-6\end{array}\right.+B\left|\begin{array}{l}x^3\\+x^2\\-6x\\ \end{array}\right.+C\left|\begin{array}{l}x^3\\+4x^2\\+3x\\ \end{array}\right.+D\left|\begin{array}{l}x^3\\-x^2\\-2x\\ \end{array}\right.$$

Igualamos los coeficientes de las mismas potencias de x:

$$\left.\begin{array}{lr}A+B+C+D & =0\\ 2A+B+4C-D & =3\\ -5A-6B+3C-2D & =-2\\ -6A & =1\end{array}\right|\ \therefore\quad \begin{array}{l}A=-\dfrac{1}{6}\\ B=1\\ C=\dfrac{3}{10}\\ D=\dfrac{17}{15}\end{array}$$

$$\therefore\ y=\frac{1}{6}\,Lx+L(x+1)+\frac{3}{10}\,L\,(x-2)-\frac{17}{15}\,L\,(x+3)$$

$$=L\,\frac{(x+1)\,(x-2)^{\frac{3}{10}}}{\sqrt[6]{x}+\sqrt[15]{(x+3)^{17}}}$$

(Timmermans, 262)

23. $$\frac{1}{x^5+x^4+2x^3+2x^2+x+1}=\frac{A(x-1)}{(x^2+1)^2}+\frac{B(x-1)}{x^2+1}.$$

$$\therefore\ y=\frac{1}{4}\cdot\frac{x+1}{x^2+1}+\frac{1}{2}\operatorname{arc\,tg} x+\frac{1}{4}L\frac{x+1}{\sqrt{x^2+1}}$$

(Grégory, 263)

24. $$\frac{x^3+x^2+2}{x^5-2x^3+x}dx$$

$$=\frac{2dx}{x}+\frac{dx}{(x-1)^2}-\frac{\frac{3}{4}dx}{x-1}-\frac{\frac{1}{2}dx}{(x+1)^2}-\frac{\frac{5}{4}dx}{x+1}$$

$$\therefore\ y=2Lx-\frac{1}{x-1}-\frac{3}{4}L(x-1)+\frac{1}{2(x+1)}-\frac{5}{4}L(x+1).$$

25. $$\frac{2x^2+1}{(x-2)^3(x+3)^2}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x+2)^3}$$

$$+\frac{D}{x+3}+\frac{E}{(x+3)^2}$$

$$\therefore\ 2x^2+1=A(x-2)^2(x+3)^2$$

$$+B(x-2)(x+3)^2$$

$$+C(x+3)^2$$

$$+D(x-2)^3(x+3)$$

$$+E(x-2)^3.$$

Sea

$$x = 2, \text{ se obtiene: } 8 + 1 = C(2+3)^2 \therefore C = \frac{9}{25}.$$

Sea

$$x = -3, \text{ se obtiene: } +18 + 1 = E.-5^3 \therefore E = -\frac{19}{125}.$$

Para encontrar los demas coeficientes, diferenciamos los dos miembros:

$$4x = A\,[2(x-2)(x+3)^2 + 2(x-2)^2(x+3)]$$

$$+ B\,[(x+3)^2 + 2(x-2)(x+3)]$$

$$+ C.2(x+3)$$

$$+ D\,[3(x-2)^2(x+3) + (x-2)^3]$$

$$+ E.3(x-2)^2.$$

Sea, ahora,

$$x = 2:\ 8 = 25\,B + 10\,E = 25\,B + \frac{90}{25}$$

$$\therefore B = \frac{22}{125}.$$

Sea

$$x = -3:\ -12 = D(-5)^3 + E.3.25 = -125\,E - \frac{57}{5}$$

$$\therefore\ D = \frac{3}{6250}$$

Diferenciemos de nuevo, conservando sólo los términos que no se anulan para $x=2$:

$$4 = A.2\,(x+3)^2 + B\,[2\,(x+3)+2\,(x+3)] + C.2.$$

Sea

$$x=2:\ 4=50\,A+20\,B+2\,C\ \therefore\ A = -\frac{3}{625}$$

$$\therefore\ y = -\frac{3}{5^4(x-2)} + \frac{22}{5^3(x-2)^2} + \frac{9}{25\,(x-2)^3}$$

$$+ \frac{3}{5^4\,(x+3)} - \frac{19}{5^3\,(x+3)^2}.$$

(Homersham Cox, 55)

26. $$\frac{x^3-x^2+x-1}{(x-1)^3(x+2)^2(x+1)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

$$+ \frac{D}{(x+2)^2} + \frac{B}{x+2} + \frac{F}{x+1}$$

$$\therefore\ x^3 - x^2 + x - 1 = A\,(x+2)^2(x+1)$$

$$+ B\,(x-1)\,(x+2)^2(x+1)$$

$$+ C\,(x-1)^2(x+2)^2(x+1)$$

$$+ D\,(x-1)^3(x+1)$$

$$+ E (x-1)^3 (x+2) (x+1)$$

$$+ F (x-1)^3 (x+2)^2$$

Sea

$x=-1$, se obtiene $F = \frac{1}{2}$; para $x = -2$, resulta $D = \frac{5}{9}$

Diferenciando, se determinan los demas coeficientes

27. $y = \int \frac{dt}{(1-t^2)^2}$ Aquí $(1-t^2)^2 = [(1+t)(1-t)]^2$

$$\frac{1}{(1-t^2)^2} + \frac{A}{1+t} + \frac{B}{1-t} + \frac{C}{(1+t)^2} + \frac{D}{(1-t)^2}$$

$$\therefore\ 1 = A (1-t)^2 (1+t) + B (1+t)^2 (1-t)$$

$$+ C (1-t)^2 + D (1+t)^2 .$$

Sea

$$x = 1, \qquad D (1+1)^2 = 1 \qquad \therefore\ D = \frac{1}{4}$$

Sea

$$x = -1, \quad C (1+1)^2 = 1 \qquad \therefore\ C = \frac{1}{4}$$

Para encontrar los demas coeficientes, diferenciamos:

$$0 = A (1-t)^2 - 2 A (1+t) (1-t) - B (1+t)^2$$

$$+2B(1-t)(1+t)-2C(1-t)+2D(1+t).$$

Sea

$$x=1; \qquad -B(1+1)^2+2D(1+1)=0 \quad \therefore B=\frac{1}{4}$$

Sea

$$x=-1, \qquad A(1+1)^2-2C(1+1)=0 \quad \therefore A=\frac{1}{4}$$

$$\therefore y=\frac{1}{4}\int\left[\frac{1}{1+t}+\frac{1}{1-t}+\frac{1}{(1+t)^2}+\frac{1}{(1-t)^2}\right]$$

$$=\frac{1}{4}\left[L(1+t)-L(1-t)+\frac{1}{1-t}-\frac{1}{1+t}\right]$$

$$=\frac{1}{4}\left[L\frac{1+t}{1-t}+\frac{2t}{1-t^2}\right]$$

28. $y=\int\frac{dx}{1-x^6}$ (Euler, I, 40)

$$\frac{dx}{1-x^6}=\frac{A}{1+x}+\frac{B}{1-x}+\frac{C+Dx}{1+x+x^2}+\frac{E+Fx}{1-x+x^2}$$

Hagamos desaparecer los denominadores.

$$1=A(1-x)(1+x+x^2)(1-x+x^2)$$

$$+B(1+x)(1+x+x^2)(1-x+x^2)$$

$$+(C+Dx)(1-x^2)(1-x+x^2)$$

$$+(E+Fx)(1-x^2)(1+x+x^2)$$

Sea

$$x=1, \quad 1=B(1+1)(1+1+1)(1-1+1) \therefore B=\frac{1}{6}$$

$$x=-1, \quad 1=A(1+1)(1-1+1)(1+1+1) \therefore A=\frac{1}{6}$$

Procediendo como en los ejercicios anteriores encontramos que

$$\frac{1}{1-x^6}=\frac{\frac{1}{2}}{1+x}+\frac{\frac{1}{6}}{1-x}+\frac{\frac{1}{3}+\frac{1}{6}x}{1+x+x^2}+\frac{\frac{1}{3}+\frac{1}{6}x}{1-x+x^2}.$$

$$\therefore y=\frac{1}{6}L\frac{1+x}{1-x}\sqrt{\frac{1+x+x^2}{1-x+x^2}}+\frac{1}{2\sqrt{3}}\text{ arc tg }\frac{\sqrt{3}x}{1-x^2}$$

(Grégory, 264)

$$=-\frac{1}{6}L(1-x)-\frac{1}{12}L(1-x+x^2)+\frac{1}{2\sqrt{3}}\text{arc tg }\frac{x\sqrt{3}}{2-x}$$

$$+\frac{1}{6}L(1+x)+\frac{1}{12}L(1+x+x^2)+\frac{1}{2\sqrt{3}}\text{arc tg }\frac{x\sqrt{3}}{2+x}$$

(Peacock, 279)

29. $$\frac{1}{x^n(x-1)^n}=\frac{1}{x^n}+\frac{1}{(1-x)^n}+n\left[\frac{1}{x^{n-1}}+\frac{1}{(1-x)^{n-1}}\right]$$

$$+\frac{n(n+1)}{2!}\left[\frac{1}{x^{n-2}}+\frac{1}{(1-x)^{n-2}}\right]+\ldots\ldots$$

$$\frac{n(n+1)(n+2)\ldots 2(n-1)}{(n-1)!}\left[\frac{1}{x}+\frac{1}{1-x}\right]$$

$$\therefore y=\frac{1}{n-1}\left[\frac{1}{(1-x)^{n-1}}-\frac{n}{x^{n-1}}\right]+\frac{n}{n-2}\left[\frac{n}{(1-x)^{n-2}}-\frac{1}{x^{n-2}}\right]$$

$$+\ldots\ldots+\frac{n(n+1)\ldots 2(n-1)}{(n-1)!}\,L\,\frac{1}{1-x}$$

(Murphy, VI)

54. *Descomposicion de radicales.*—Para descomponer en un binomio el radical $\sqrt{1+x^2}$, se multiplica por $1=\sqrt{1+x^2}:\sqrt{1+x^2}$:

$$\sqrt{1+x^2}=\frac{1+x^2}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}+\frac{x^2}{\sqrt{1+x^2}}.$$

De igual manera se obtiene:

$$\sqrt{a^2-x^2}=\frac{a^2}{\sqrt{a^2-x^2}}-\frac{x^2}{\sqrt{a^2-x^2}},$$

$$\sqrt{a+bx+cx^2}=\frac{a}{\sqrt{a+bx+cx^2}}+\frac{b}{\sqrt{a+bx+cx^2}}$$

$$+\frac{cx^2}{\sqrt{a+bx+cx^2}}.$$

En la integracion por partes se hará principalmente esta descomposicion.

$$1. \quad \int_x \sqrt{\frac{1+x}{1-x}} = \int_x \frac{1+x}{\sqrt{1-x^2}} = \int_x \left(\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} \right)$$

La primera integral es conocida, i es igual a

$$\text{arc sen } x.$$

La segunda se obtiene asi:

$$\int_x \frac{x}{\sqrt{1-x^2}} = -\frac{1}{2} \int \left(1-x^2\right)^{-\frac{1}{2}} . -2\,dx = -\sqrt{1-x^2}$$

$$\therefore \int \sqrt{\frac{1x+}{1-x}} \text{ arc sen } x - \sqrt{1-x^2}$$

(Euler, IV, 10)

$$2. \quad \int_x \frac{\sqrt{a-x}}{\sqrt{a+x}} = \int_x \frac{a}{\sqrt{a^2-x^2}} - \int_x \frac{x}{\sqrt{a^2-x^2}}$$

$$= a \text{ arc sen } \frac{x}{a} + \sqrt{a^2-x^2}$$

$$3. \quad \int \frac{dx}{\sqrt{x+a} + \sqrt{x+b}} = \int_x \frac{\sqrt{x+a} - \sqrt{x+b}}{x+a-(x+b)}$$

$$= \frac{1}{a-b} \int_x \left(\sqrt{x+a} \right) - (x+b)$$

$$= \frac{2}{3\,(a-b)} \left[(x+b)^{\frac{2}{3}} - (x+b)^{\frac{3}{2}} \right]$$

(Grégory, 252)

4. $$\int \frac{\sqrt{x+a}}{x\sqrt{x-a}}\, d\,x = \int \frac{d\,x}{\sqrt{x^2-a^2}} + \int \frac{a\,d\,x}{x\sqrt{x^2-a^2}}.$$

$$= \text{arc sec}\, \frac{x}{a} + L\, \frac{x + \sqrt{x^2-a^2}}{a}$$

5. $$\int \frac{\sqrt{x^2-a^2}}{x} = \int_x \frac{x}{\sqrt{x^2-a^2}} - \int_x \frac{a^2}{\sqrt{x^2-a^2}}$$

$$= \sqrt{x^2-a^2} - a.\,\text{arc sec}\, \frac{x}{a}$$

6. $$\int \frac{\sqrt{x^2+a^2}}{x}\, d\,x = \sqrt{x^2+a^2} + a\,L\, \frac{x}{\sqrt{x^2+a^2}+a}$$

7. $$\int \frac{\sqrt{x+1}}{x\sqrt{x-1}}\, d\,x = \int \frac{d\,x}{\sqrt{x^2-1}} + \frac{d\,x}{\sqrt{x^2-1}}$$

$$= L\,(x + \sqrt{x^2-1}) + \text{arc sec}\, x.$$

(Brahy, 20)

55. *Descomposicion trigonométrica*, o sea descomponer la derivada trigonométrica en dos o mas términos.

1. $\int \text{versen}\, x\, d\,x = \int (1 - \cos x)\, d\,x$

$$= \int dx - \int \cos x \, dx = x - \operatorname{sen} x$$

2. $\int \operatorname{convers\,en} x \, dx = \int (1 - \operatorname{sen} x) \, dx = x + \cos x.$

3. $\int \operatorname{sen}^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx$

$$= \frac{1}{2} \int dx - \frac{1}{4} \int \cos 2x \, dx$$

$$= \frac{1}{2} x - \frac{1}{4} \operatorname{sen} 2x$$

4. $\int \cos^2 x \, dx = \int \frac{1 + \cos 2x}{2} \, dx = \frac{1}{2} x + \frac{1}{4} \operatorname{sen} 2x.$

5. Otro método. (Humbert, I, 199).

$$\int_x \cos^2 x + \int_x \operatorname{sen}^2 x = \int_x (\cos^2 x + \operatorname{sen} x) = \int dx = x,$$

$$\int_x \cos^2 x - \int_x \operatorname{sen}^2 x = \int_x (\cos^2 x - \operatorname{sen}^2 x) = \int_x \cos 2x = \frac{1}{2} \operatorname{sen}^2 x,$$

sumando i restando, se obtiene:

$$\int \cos^2 x \, dx = \frac{1}{2} x + \frac{1}{4} \operatorname{sen}^2 x \qquad \text{(Dariès, 116)}$$

$$\int \operatorname{sen}^2 x \, dx = \frac{1}{2} x - \frac{1}{3} \operatorname{sen}^2 x \qquad \text{(Bowser, 246)}$$

6. $\int tg^2 x\, d\, x = \int \frac{sen^2 x}{cos^2 x}\, d\, x = \int \frac{1 - cos^2 x}{cos^2 x}\, x\, d$

$$= \int \frac{d\, x}{cos^2 x} - \int d\, x = tg\, x - x.$$

(Sonnet, 232)

7. Otro método. (Frenet, 252).

$$\int tg^2 x\, d\, x = \int (sec^2 x - 1)\, d\, x = tg\, x - x.$$

8. $\int cot^2 x\, d\, x = \int_x \frac{cos^2 x}{sen^2 x} = \int_x \frac{1 - sen^2 x}{sen^2 x}$

$$= - cot\, x - x.$$

9. Otro método.

$$\int_x cot^2 x = \int_x (cosec^2 x - 1) = - cot\, x - x.$$

10. $\int_{mx} sen^2\, m\, x = \frac{1}{m} \int (1 - cos^2 m\, x)\, d\, m\, x$

$$= \frac{1}{4\, m} (2\, m\, x - sen\, 2\, m\, x)$$

(Perry, 364)

11. $\int cos^2 n\, x\, d\, x = \frac{1}{n} \int cos^2 n\, x\, d\, n\, x$

$$= \frac{1}{4\,n}\,(2\,n\,x + \text{sen}^2 n\,x)$$

12. $$\int_x \text{sen}^3 x = \frac{1}{4} \int (3\,\text{sen}\,x - \text{sen}\,3\,x)$$

$$= -\frac{1}{4}\,(3\cos x + \frac{1}{3}\cos 3\,x)$$

$$= -\frac{1}{3}\,\text{sen}^2 x \cos x - \frac{2}{3}\cos x$$

(Peacock, 311)

13. $$\int_x \sec^4 x = \int \sec^2 x \sec^2 x\, d\,x = \int (1 + \text{tg}^2 x)\, d\,\text{tg}\,x$$

$$= \text{tg}\,x + \frac{1}{3}\,\text{tg}^3 x.$$ (Roberts, 8)

14. $$\int_x \text{sen}^5 x = \frac{1}{16} \int (\text{sen}\,5\,x - 5\,\text{sen}\,3\,x + 10\,\text{sen}\,x)$$

$$= \frac{1}{16}\left(-\frac{1}{5}\cos 5\,x + \frac{5}{3}\cos 3\,x - 10\cos x\right)$$

15. $$\int \frac{1}{\text{sen}\,x}\, d\,x.$$ (Jarrett, 124).

$$\frac{1}{\text{sen}\,x} = \frac{\text{sen}\,x}{\text{sen}^2 x} = \frac{\text{sen}\,x}{1 - \cos^2 x}.$$

Descompongamos esta fraccion en fracciones parciales:

$$\therefore \int \frac{1}{\text{sen } x} d\,x = \frac{1}{2} \int \left(\frac{d - \cos x}{1 - \cos x} - \frac{d \cos x}{1 + \cos x} \right)$$

$$= \frac{1}{2} \, L \, \frac{1 - \cos x}{7 + \cos x} = L \, \text{tg} \, \frac{1}{2} x \, .$$

16. $$\int \frac{1}{\cos x} d\,x = \frac{1}{2} \int \left(\frac{d \text{sen } x}{1 + \text{sen } x} - \frac{d - \text{sen } x}{1 - \text{sen } x} \right)$$

$$= \frac{1}{2} \Big[L\,(1 + \text{sen } x) - L\,(1 - \text{sen } x) \Big]$$

$$= L \sqrt{\frac{1 + \text{sen } x}{1 - \text{sen } x}} = L \cot \left(\frac{1}{4} \pi - \frac{1}{2} x \right)$$

(Id., id.)

17. $$\int \frac{1}{\text{sen } x} d\,x = \int \frac{\text{sen}^2 \frac{1}{2} x + \cos^2 \frac{1}{2} x}{4 \, \text{sen}^2 \frac{1}{2} x \cos^2 \frac{1}{2} x}$$

$$= \frac{1}{2} \int \frac{\frac{1}{2} d\,x}{\cos^2 \frac{1}{2} x} + \frac{1}{2} \int \frac{\frac{1}{2} d\,x}{\text{sen}^2 \frac{1}{2} x}$$

$$= \frac{1}{2} \left(\text{tg} \, \frac{1}{2} x - \cot \frac{1}{2} x \right)$$

(De Longchamps, 643)

18. $$\int \frac{d\,x}{\text{sen } x \cos x} = \int \frac{\text{sen}^2 x + \cos^2 x}{\text{sen } x \cos x} d\,x$$

$$= \int \frac{\text{sen } x}{\cos x} \, d\,x + \int \frac{\cos x}{\text{sen } x} \, d\,x$$

$$= -\text{L} \cos x + \text{L sen } x = \text{L tg } x.$$

(Timmermans, 252)

19. $$\int \frac{d\,x}{\text{sen}^2 x \cos^2 x} = \int \frac{\text{sen}^2 x + \cos^2 x}{\text{sen}^2 x \cos^2 x} \, d\,x$$

$$= \int \frac{d\,x}{\cos^2 x} + \int \frac{d\,x}{\text{sen}^2 x} = \text{tg } x - \cot x$$

(J. Bertrand, 7)

20. $$\int \frac{\text{sen } x \cos^2 x}{1 + a^2 \cos^2 x} \, d\,x = \frac{1}{a^2} \int \left(\text{sen } x \, d\,x - \frac{\text{sen } x \, d\,x}{1 + a^2 \cos^2 x} \right)$$

$$= -\frac{1}{a^2} \cos x - \frac{1}{a^3} \text{ arc tg } (a \cos x)$$

21. $y = \int \frac{1}{a + b \text{ tg } x} \, d\,x$. Multipliquemos por $\cos x$ i agreguemos en seguida

$$\frac{a}{a^2 + b^2} - \frac{a}{a^2 + b^2}:$$

$$\frac{1}{a + b \text{ tg } x} = \frac{\cos x}{a \cos x + b \text{ sen } x} + \frac{a}{a^2 + b^2} - \frac{a}{a^2 + b^2}$$

$$= \frac{a^2 \cos x + b^2 \cos x - a^2 \cos x - ab \operatorname{sen} x}{(a \cos x + b \operatorname{sen} x)(a^2 + b^2)} + \frac{a}{a^2 + b^2}$$

$$\therefore y = \frac{b}{a^2 + b^2} \int \frac{d\, a \cos x + d\, b \operatorname{sen} x}{a \cos x + b \operatorname{sen} x} + \int \frac{a}{a^2 + b^2} d\, x$$

$$= \frac{b}{a^2 + b^2} L(a \cos x + b \operatorname{sen} x) + \frac{a\, x}{a^2 + b^2}$$

22. $\int \operatorname{sen} p\, x \cos q\, x\, d\, x$

$$= \frac{1}{2} \int \operatorname{sen} (p + q)\, x\, d\, x + \frac{1}{2} \int \operatorname{sen} (p - q)\, x\, d\, x$$

$$= -\frac{1}{2} \left[\frac{\cos (p + q)\, x}{p + q} + \frac{\cos (p + q)\, x}{p - q} \right]$$

22. $\int \cos p\, x \cos q\, x\, d\, x = \frac{1}{2} \left[\frac{\operatorname{sen} (p + q) x}{p + q} + \frac{\operatorname{sen} (p - q)\, x}{p - q} \right]$

23. $\int \cos (a\, x + b) \cos (a^1 x + b^1)\, d\, x$

$$= \frac{\operatorname{sen} [(a + a^1)\, x + b + b^1]}{a\,(a + a^1)}$$

$$+ \frac{\operatorname{sen} [(a - a^1)\, x + b - b^1)]}{2\,(a - a^1)}$$

(Tannery, 511)

24. $\int \operatorname{sen} (a\, x + b) \operatorname{sen} (a^1 x + b^1)\, d\, x$

$$= \frac{\text{sen}\left[(a-a^1)x+b-b^1\right]}{2(a-a^1)} - \frac{\text{sen}\left[(a+a^1)x+b+b^1\right]}{2(a+a^1)}$$

(Sturm, 359)

25. $\int \cos x \cos 2x \cos 3x \, dx$. Apliquemos la fórmula

$$\cos a \cos b = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$$

$$\therefore \cos x \cos 2x = \frac{1}{2}\cos 3x + \frac{1}{2}\cos x$$

$$\therefore \cos x + \cos 2x + \cos 3x = \frac{1}{2}\cos^2 3x + \frac{1}{2}\cos x \cos 3x:$$

pero $\frac{1}{2}\cos^2 3x = 1 + \cos 6x$

y $\cos x \cos 3x = \frac{1}{2}\cos 4x + \frac{1}{2}\cos^2 x.$

$$\therefore y = \frac{1}{4}\left[\frac{\text{sen } 6x}{6} + \frac{\text{sen } 4x}{4} + \frac{\text{sen } 2x}{2} + x\right]$$

(De Comberousse, 755)

26. $\int \text{sen } x \text{ sen } 2x \text{ sen } 3x \, dx.$

$$\text{sen } a \text{ sen } b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

$$\therefore \text{ sen } x \text{ sen } 3\,x = \frac{1}{2} \cos^2 x - \frac{1}{2} \cos 4\,x$$

$$\therefore y = \frac{1}{4}\left[x + \frac{\text{sen } 2\,x}{2} - \frac{\text{sen } 4\,x}{4} - \frac{\text{sen } 6\,x}{6} \right].$$

27. $\int (\text{tg}^2 x - \cot^2 x) d\,x = \text{tg}\,x + \cot x = \frac{2}{\text{sen } 2\,x}.$

56. *Integracion por serie,* o sea desarrollar la derivada en serie i aplicar en seguida la regla de la suma (Núm. 31).

1. $\int (a + b\,x)^n x\,d\,x$. Desarrollemos la potencia segun el binomio de Newton:

$$(a + b\,x)^n = a^n + n\,a^{n-1}\,b\,x + \frac{n\,(n-1)}{2!}\,a^{n-2}\,b^2\,x^2$$

$$+ \frac{n\,(n-1)\,(n-2)}{3\,!}\,a^{n-3}\,b^3\,x^3 + \ldots\ldots$$

Multipliquemos por $x\,d\,x$ e integremos cada término:

$$y = \frac{1}{2}\,a^n\,x^2 + \frac{1}{3}\,n\,a^{n-1}\,b\,x^3 + \frac{1}{4}\,\frac{n\,(n-1)}{2!}\,a^{n-2}\,b^2\,x^4$$

$$+ \ldots\ldots$$

2. $\int x^m\,(a + b\,x^n)^{\frac{p}{q}}\,d\,x$. Es preferible desarrollar así:

$$a + b\,x^n = a\left(1 + \frac{b}{a}\,x^n\right)$$

$$\therefore (a + b x)^{\frac{p}{q}} = a^{\frac{p}{q}}\left(1 + \frac{b}{a} x^n\right)^{\frac{p}{p}}$$

$$= a^{\frac{p}{q}}\left[1 + \frac{p}{q} \cdot \frac{b}{a} x^n + \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)}{2!} \frac{b^2}{a^2} x^{2n}\right.$$

$$\left. + \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)\left(\frac{p}{q} - 2\right)}{3!} \frac{b^3}{a^3} x^{3n} + \ldots\ldots\right]$$

Multiplicamos por $x^m\, d\, x$ i en seguida integramos:

$$y = a^{\frac{p}{q}}\left[\frac{x^{m+1}}{m+1} + \frac{p}{q} \cdot \frac{b}{a} \cdot \frac{x^{m+n+1}}{m+n+1}\right.$$

$$\left. + \frac{\frac{p}{q}\left(\frac{p}{q} - 1\right)}{2!} \frac{b^2}{a^2} \frac{x^{m+2n+1}}{m+2n+1} + \ldots\ldots\right]$$

(Timmermans, 385)

3. $\int \frac{1}{1+x} d\, x$. *I Método*, por division: efectuemos la division i encontraremos que

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots\ldots \pm x^n \mp \ldots\ldots$$

Multipliquemos por $d\, x$ e integremos:

$$\int \frac{dx}{1+x} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \pm \frac{x^{n+1}}{n+1} \mp \ldots\ldots$$

II Método, por el binomio:

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + \frac{-1 \,.\, -2}{2!} x^2$$

$$+ \frac{-1 \,.\, -2 \,.\, -3}{3!} x^3 - \ldots\ldots$$

III Método, por los coeficientes indeterminados:

Supongamos que el desarrollo es,

$$\frac{1}{1+x} = A + B\,x + C\,x^2 + D\,x^3 + E\,x^4 + \ldots\ldots$$

Hagamos desaparecer el denominador:

$$1 = \quad A + B\,x + C\,x^2 + D\,x^3 + E\,x^4 + \ldots$$

$$\frac{\qquad + A\,x + B\,x^2 + C\,x^3 + D\,x^4 + \ldots}{1 = \quad A + (A+B)\,x + (B+C)\,x^2 + (C+D)\,x^3 + \ldots}$$

Igualemos dos coeficientes de las mismas potencias de x o sea $x = 0$ (Véase el núm. 53):

$$A = 1,\ A + B = 0,\ B + C = 0,\ C + D = 0, \ldots\ldots$$

$$\therefore\ A = 1,\ B = -1,\ C = 1,\ D = -1, \ldots\ldots$$

Sustituyendo, encontramos el mismo desarrollo.

Observacion importante. Hemos visto que

$$\int \frac{dx}{1+x} = \int \frac{d(1+x)}{1+x} = L(1+x)$$

luego, igualando con el resultado de mas arriba, llegamos al *desarrollo en serie* de un logaritmo:

$$L(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 + \ldots\ldots$$

4. $$\int \frac{1}{a+x}\,dx = \frac{1}{a}x - \frac{1}{2}\cdot\frac{x^2}{a^2} + \frac{1}{3}\cdot\frac{x^3}{a^3} + \ldots$$

(Tedenat, II, 245)

5. $$\int \frac{a\,dx}{a-x} = a\,x + \frac{1}{2}x^2 + \frac{1}{3}\cdot\frac{x^3}{a} + \frac{1}{4}\cdot\frac{x^4}{a^2} + \ldots$$

6. $$\int \frac{dx}{1+x^2} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$$

(Boucharlat, 156)

Por otra parte, sabemos que

$$\int \frac{dx}{1+x^2} = \text{arc}\,.\,\text{tg}\,x.$$

Hacemos arc tg $x = \frac{1}{4}\pi$.·. $x = 1$, i obtendremos la de

Leibnitz:

$$\frac{1}{4}\pi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

7. $$\int \frac{a^2}{a^2+x^2} x = \int \left(1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \ldots\right) dx$$

$$= x - \frac{1}{3}\frac{x^3}{a_2} + \frac{1}{5}\frac{x^5}{a^4} - \ldots$$

$$\therefore a \text{ arc tg } \frac{x}{a} = x - \frac{1}{3}\frac{x^3}{a^2} + \frac{1}{5}\frac{x^5}{a^4} - \ldots$$

Esta es la série de Santiago Grégory, (1671)

Haciendo $x = \frac{1}{3}\sqrt{3}$ se obtiene la del Dr. Halley.

(Colin Mac Laurin, 604)

8. $$\int_x \sqrt{1-x^2} = x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \frac{1}{112}x^7 - \ldots\ldots$$

(Millar, 37)

9. $$\int_x \sqrt{a^2 \div x^2} = a\,x + \frac{x^3}{6a} + \frac{x^5}{4a^3} + \ldots$$

(Stone, 23)

10. $$\int_x \sqrt{a^2 - x^2} = a\,x - \frac{x^3}{6a} - \frac{x^3}{4a^3} - \frac{x^7}{112a^5}$$

$$-\frac{5\,x^9}{1152\,a^7}-\ldots$$

11. $$\int_x \frac{1}{\sqrt{1-x^2}} = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1.3}{2.4}\,\frac{x^5}{5}$$

$$+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\ldots$$

$$\therefore \text{arc sen}\, x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{1}{164}x^7 + \ldots$$

Si hacemos $x=1$ se obtiene la serie de Newton.

(Catalan, 603)

12. $$\int \frac{dx}{\sqrt{1-x^4}} = x + \frac{1}{10}x^5 + \frac{3}{72}x^9 + \frac{15}{624}x^{13} + \ldots$$

13. $\int \frac{a^x}{x}\,dx$. Desarrollemos la funcion esponencial por la fórmula de Mac Laurin:

$$a^x = 1 + \frac{x\,L\,a}{1!} + \frac{x^2\,L^2\,a}{2!} + \frac{x^3\,L^3\,a}{3!} + \ldots$$

Multipliquemos por $\frac{dx}{x}$ e integramos:

$$\int \frac{a^x}{x}\,dx = L\,x + \frac{L\,a}{1!}\,x + \frac{L^2\,a}{2!\,3}\,x^3 + \frac{L^3\,a}{3!\,4}\,x^4 + \ldots$$

(Pauly, 201)

14. $\int \frac{e^x}{x} dx = Lx + x + \frac{x^2}{2!2} + \frac{x^3}{3!3} + \ldots$

CAPITULO V

INTEGRACION POR SUSTITUCION

57. *Carácter jeneral de este método.*—En la mayor parte de las integrales tratadas hasta aquí, este método está implícitamente empleado o mediante su ayuda se puede hacer la integracion.

En la misma integracion por partes encontramos la sustitucion de la variable.

Sin embargo, no es necesario acudir a este método a cada paso, porque ademas de las dificultades que presenta su correcto empleo, por lo jeneral prolonga los cálculos.

Consiste, como ya se dijo en el número 23, en un *cambio de la variable* o en representar una funcion compuesta por una variable o funcion *auxiliar*, que a su vez puede ser simple o compuesta, monomia, polinomia, inversa, producto, etc.

Por ejemplo, sea

$$dy = f(x)\,dx.$$

Para eliminar la variable independiente, hacemos

$$x = \phi(z) \quad \therefore \quad dx = \phi'(z)\,dz,$$

i sustituimos:

$$dy = f[\phi z]\,\phi' z\,dz.$$

La nueva forma ha de ser una integral conocida o de fácil integracion.

58. *Auxiliar monomia.*— Llamamos así a la auxiliar simple z, a z o a + z.

1. $y = \int (a + b\,x)^2\,dx$. Sea $a + b\,x = z$ ∴ $dx = \frac{1}{2}\,dz$.

sustituimos:

$$y = \int z^2 \cdot \frac{1}{b}\,dz = \frac{1}{b}\int z^2\,dz = \frac{1}{b} \cdot \frac{1}{3} z^3$$

sustituimos el binomio:

$$y = \frac{1}{b} \cdot \frac{1}{3}\,(a + b\,x)^3$$

Se obtiene mas fácilmente la integral introduciendo el coeficiente b.

2. $y = \int (a\,x + b)^n dx$. $a\,x + b = z$ ∴ $dx = \frac{dz}{a}$

$$y = \frac{1}{a}\int z^n\,dz = \frac{1}{a} \cdot \frac{1}{n+1}\,(a\,x + b)^{n+1}$$

(Haag, 108)

3. $y = \int (a\,x - b)^p\,dx$ $a\,x - b = z$ ∴ $dx = \frac{dz}{a}$

$$y = \frac{1}{a}\int z^p\,dz = \frac{1}{a(p+1)}\,(a\,x + b)^{p+1}$$

(Timmermans, 247)

4. $y = \int (a x^n + b)^m = \frac{1}{a n} \int z^m b z = \frac{(a x^n + b)^{m+1}}{a n (m+1)}$

(Lacroix, II, 4)

5. $y = \int (a + b x^n)^{\frac{p}{q}} x^{m-1} d x.$

Sea

$$a + b x^n = z \therefore d x = \frac{1}{b n} x^{1-n} d z,$$

$$x^n = \frac{z-a}{b}; \quad x = \left(\frac{z-a}{b}\right)^{\frac{1}{n}}, \quad x^{n-1} = \left(\frac{z-a}{b}\right)^{\frac{1-n}{n}}$$

$$\therefore y = \int z^{\frac{p}{q}} \left(\frac{z-a}{b}\right)^{\frac{m-1}{n}} \cdot \frac{1}{b n} \left(\frac{z-a}{b}\right)^{\frac{1-n}{n}} d z$$

$$= \frac{1}{n b^{\frac{m}{n}}} \int z^{\frac{p}{q}} (z-a)^{\frac{m-n}{n}} d z.$$

(Roberts, 9)

La integracion se termina desarrollando el binomio.

6. $y = \int \frac{a}{1 + a^2 x^2} d x.$ Sea $x = \frac{v}{a} \therefore d x = \frac{d v}{a}$

$$y = \int \frac{d v}{1 + v^2} = \text{arc tg } a x$$

(Pauly, 158)

7. $$y = \int \frac{dx}{1-(a+x)^2} = \int \frac{dz}{\sqrt{1-z^2}} = \text{arc sen } (a+x)$$

(De Freycinet, 108)

8. $$y = \int \frac{x\,dx}{a^2+x^2} \cdot \text{ Sea } a^2+x^2 = z \;\therefore\; x\,dx = \frac{1}{2}\,dz$$

$$y = \frac{1}{2}\int \frac{dz}{z} = \frac{1}{2}\,L\,z = L\,\sqrt{a^2+x^2}$$

(Cournot, II, 2)

9. $$y = \int \frac{dx}{x^2+p\,x+q} \cdot \text{ Sea } x + \frac{1}{2}\,p = t\left(q - \frac{1}{4}\,p^2\right)^{\frac{1}{2}}$$

$$\therefore\; dx = dt\,\sqrt{q - \frac{1}{4}\,p^2}$$

$$y = \int \frac{\left(q - \frac{1}{4}\,p^2\right)^{\frac{1}{2}}}{\left(q - \frac{1}{4}\,p^2\right)t^2 + \left(q - \frac{1}{4}\,p^2\right)}\,dt$$

$$= \frac{1}{\sqrt{q - \frac{1}{4}\,p^2}}\int \frac{dt}{1+t^2} = \frac{1}{\sqrt{q - \frac{1}{4}\,p^2}}\,\text{arc tg}\,\frac{x + \frac{1}{2}\,p}{\sqrt{q - \frac{1}{4}\,p^2}}$$

(Serret, II, 16)

10. $$\int \frac{dx}{a+2\,b\,x+c\,x^2} = \int \frac{c\,dx}{(c\,x+b)^2+ac-b^2}\,.$$

Sea

$$c\,x + b = z \quad \therefore \quad d\,x = \frac{1}{c}\,d\,z$$

$$y = \int \frac{d\,z}{a\,c - b^2 + z^2} = \frac{1}{\sqrt{a\,c - b^2}} \text{ arc tg } \frac{c\,x + b}{\sqrt{a\,c - b^2}}$$

$$\text{o} \qquad y = \frac{1}{\sqrt{b^2 - a\,c}} \, L \, \frac{c\,x + b - \sqrt{b^2 - a\,c}}{c\,x + b + \sqrt{b^2 - a\,c}}$$

(Roberts, 11)

11. $y = \int \frac{d\,x}{(x - a)\,(x - b)}$. Sea $x = \frac{1}{2}(a + b) + z \therefore d\,x = d\,z$,

$$x - a = z + \frac{1}{2}(b - a)$$

$$x - b = z - \frac{1}{2}(b - a)$$

$$\therefore (x - a)\,(x - b) = z^2 - \frac{1}{4}(b - a)^2$$

$$y = \int \frac{d\,z}{z^2 - \frac{1}{4}(b - a)^2} = \frac{1}{a - b}\, L \, \frac{x - a}{x - b}$$

12. $y = \int \frac{5\,x^3}{3\,{}^4x + 7}$ Sea $3\,x^4 + 7 = t \therefore x^3 d\,x = \frac{1}{12}\,d\,t$

$$y = \frac{5}{12} \int \frac{d\,t}{t} = \frac{5}{12}\,(3\,x^4 + 7)$$

(Sturm, 317)

13. $$y = \int \frac{dx}{x(a+bx^m)} = \int \frac{x^{m-1}dx}{x^m(a+bx^m)}$$

$$= \frac{1}{m}\int \frac{dz}{z(a+bz)} = \frac{1}{am} L \frac{x^m}{a+bx^m}$$

(Jordan, II, 11)

14. $$y = \int \frac{x\,dx}{(a+bx)^3} \quad \text{Sea} \quad a+bx=z \;\therefore\; dx = \frac{1}{b}dz$$

$$y = \frac{1}{b^2}\int \left(z^{-2} - a z^{-3}\right) dz = \frac{-1}{2b^2} \cdot \frac{2bx+a}{(a+bx)^2}$$

15. $$y = \frac{x\,dx}{(a+bx)^{\frac{2}{3}}} = \frac{1}{b^2}\int \frac{z-a}{z^{\frac{2}{3}}} dz$$

$$= \frac{3}{4} \cdot \frac{1}{b^2}(bx-3a)(a+bx)^{\frac{1}{3}}$$

16. $$y = \int \frac{x\,dx}{(x^2+a^2)^n} = \frac{1}{2}\int z^{-n}dz = -\frac{1}{2(n-1)(x^2+a^2)^{n-1}}$$

(Brahy, 14)

17. $$y = \int \frac{A x^m}{(a+bx)^n} dx = \frac{A}{b}\int \frac{\left(\frac{z-a}{b}\right)}{z n} dz$$

$$= \frac{A}{b^{m+1}}\int \left(\frac{z-a}{z^n}\right)^m dz$$

Desarrollamos por el binomio e integramos cada término:

$$y = \frac{A}{b^{m+1}}\left[\frac{(a+bx)^{m-n+1}}{m-n+1} - \frac{m\,a\,(a+bx)^{m-n}}{m-n} + \frac{a^2 m(m-1)(a+bx)^{m-n-1}}{m-n-1} + \dots\right]$$

18. $y = \int \frac{dx}{(a+bx^n)^{\frac{n+1}{n}}}$ Multipliquemos los dos términos por x^{-n-1} i en seguida hacemos $a\,x^{-n} + b = z$

$$y = \frac{1}{a}\,\frac{x}{(a+bx^n)^{\frac{1}{n}}}.$$

19. $y = \int \sqrt{ax+b}\,.dx$ Sea $d\,x + b = z$ ∴ $dx = \frac{1}{a}\,dz$

$$y = \frac{1}{a}\int z^{\frac{1}{2}}\,dz = \frac{2}{3a}\sqrt{(ax+b)^3}$$

20. $y = \int \sqrt{a^2 - x^2}.\,x\,dx = -\frac{1}{2}\int z^{\frac{1}{2}}\,dz$

$$= -\frac{1}{3}(a^2 - x^2)^{\frac{3}{2}}$$

(De Morgan, 107)

21. $y = \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a\,dx}{(a^2 - a^2z^2)^{\frac{1}{2}}} = \int \frac{dz}{(1 - z^2)^{\frac{1}{2}}}$

$= \text{arc sen.}\,\frac{x}{a}$ (Pruvost, I, 550)

22. $y = \int \frac{x\,dx}{\sqrt{1-x^2}}$ · Sea $1-x^2=z \therefore x\,dx = -\frac{1}{2}dz$

$$y = -\frac{1}{2}\int z^{-\frac{1}{2}}\,dz = -\sqrt{1-x^2}$$

23. $y = \int \frac{dx}{\sqrt{2ax-x^2}}$ · Sea $x=a-z \therefore dx = -dz$

$$y = \int \frac{-dz}{\sqrt{a^2-z^2}} = \text{arc cos}\,\frac{a-x}{a} = \text{arc vers}\,\frac{x}{a}.$$

24. $y = \int \frac{dx}{\sqrt{7+6x-x^2}} = \int \frac{dx}{\sqrt{16-(x-3)^2}}$

Sea

$$x=3+4u \quad \therefore \quad dx=4\,du$$

$$y = \int \frac{du}{\sqrt{1-u^2}} = \text{arc sen}\,\frac{x-3}{4}. \qquad \text{(Sonnet, 217)}$$

25. $y = \int \frac{dx}{\sqrt{a+bx-x^2}} = \int \frac{dx}{\sqrt{a+\frac{b^2}{4}-\left(x-\frac{b}{2}\right)^2}}$

Sea

$$\frac{1}{2}b-x=t\sqrt{a+\frac{1}{4}b^2} \therefore dx = -\sqrt{a+\frac{1}{4}b^2}\,dt$$

$$y = -\int \frac{d\,t}{\sqrt{1-t^2}} = \text{arc cos}\,\frac{b-2\,x}{\sqrt{4\,a+b^2}}$$

(Sturm, 1340)

26. $$y = \int \frac{d\,x}{\sqrt{a+bx+c\,x^2}} = \frac{1}{\sqrt{c}} \int \frac{d\,x}{\sqrt{\left(x+\frac{b}{2c}\right)^2 + \frac{4ac-b^2}{4\,c^2}}}$$

Sea

$$x + \frac{b}{2\,c} = z \quad \therefore \quad d\,x = d\,z$$

$$y = \frac{1}{\sqrt{c}}\,L\left[\,2\,c\,x+b+2\,\sqrt{c}\,\sqrt{a+b\,x+6\,x^2}\right]$$

(Todhunter, 12)

27. $$y = \int \frac{3\,x-2}{\sqrt{4\,x^2+16\,x+20}}\,d\,x = \int \frac{3\,x-2}{\sqrt{4\,[(x--2)^2+1\,]}}\,d\,x$$

Sea

$$x = 2+u \quad \therefore \quad d\,x = d\,u$$

$$y = \frac{1}{2}\int \frac{u\,d\,u}{\sqrt{u^2+1}} + \int \frac{d\,u}{\sqrt{u^2+1}}$$

$$-\,2\,L\,[\,(x-2) + \sqrt{x^2-4\,x+5}\,) + \frac{3}{2}\,\sqrt{x^2-4\,x+5}\,.$$

28. $y = \int e^{a\,x}\,d\,x$. Sea $a\,x = z \;\therefore\; d\,x = \frac{1}{a}\,d\,z$

$$y = \int e^z \cdot \frac{1}{a} \, d\, z = \frac{1}{a} e^{a\,x}$$

29. $$y = \int e^{-a\,x} d\, x = -\frac{1}{a} \int e^z \, d\, z = -\frac{1}{a} e^{-a\,x}.$$

30. $$\int e^{f(x)} \, d\, f(x) = \int e^z \, d\, z = e^{f(x)}$$

(Moigno, II, 12)

31. $$y = \int \frac{e^x \, x\, d\, x}{(1+x)^2}$$ Sea $1 + x = z \;\therefore\; d\, x = d\, z$

$$y = \int \frac{e^z}{e}\left(\frac{d\, z}{z} - \frac{d\, z}{z^2}\right) = \frac{e^z}{e\, z} = \frac{e^x}{1+x}$$

(Francoeur, 374)

32. $$\int \frac{d\, x}{1+e^x}$$ Sea $1 + e^x = z \;\therefore\; e^x \, d\, x = d\, z,$

$$d\, x = \frac{d\, z}{z^2} = \frac{d\, z}{z}$$

$$y = \int \frac{d\, z}{z^2} = -\, z^{-1} = \frac{-1}{1+e^x}$$

33. $$\int \frac{d\, x}{e^x + e^{-x}}$$ Sea $e^x = y \;\therefore\; d\, x = e^{-x} d\, y = \dfrac{d\, z}{z}$

$$y = \int \frac{d\, z}{z\,(z + z^{-1})} = \int \frac{d\, z}{z\, 2 + 1} = \text{arc tg } e^x$$

(Byerly, II, 36)

34. $\int f(e^{ax})\, dx$. Sea $e^{ax} = \theta \therefore ax = L\theta$,

$$dx = \frac{d\theta}{a\theta},\ y = \frac{1}{a}\int f(\theta)\,\frac{d\theta}{\theta}$$

(Hermite, I, 255)

35. $\int \frac{Lx}{x}\, dx$. Sea $Lx = z \therefore \frac{dx}{x} = dz$

$$y = \int z\, dz = L^2 \sqrt{x}.$$

(Niewenglowski, II, 145)

36. $y = \int \frac{dx}{x\, L^n x}$ · Sea $Lx = z \therefore \frac{dx}{x} = dz$

$$y = \frac{dz}{z^n} = \frac{1}{1-n}(Lx)^{1-n}$$

37. $y = \int \sqrt{1 + Lx}\, \frac{dx}{x}$ · Sea $1 + Lx = z \therefore \frac{dx}{x} = dz$

$$y = \int \sqrt{z}\, dz = \frac{2}{3}(1 + Lx)^{\frac{3}{2}}$$

(Bertrand, 8)

38. $y = \int \frac{dx}{x} f(Lx) = \int f(z)\, dz = f\,L(x)$.

39. $y = \int \text{sen}\, ax\, dx$. Sea $ax = z \therefore dx = \frac{1}{a}\, dz$

$$y = \frac{1}{a}\int \operatorname{sen} z\, d\, z = -\frac{1}{a} \cos a\, x.$$

40. $y = \int \frac{d\,x}{\operatorname{sen} x}$ · Sea $\cos x = z$

$$\therefore \begin{cases} 1 - \cos^2 x = 1 - z\, 2, & \operatorname{sen} x = \sqrt{1 - z\, 2} \\ - \operatorname{sen} x\, d\, x = d\, z & \therefore d\, x = -\frac{d\, z}{\operatorname{sen} x} \end{cases}$$

$$y = -\int \frac{d\, z}{1 - z^2} = L \sqrt{\frac{1 - \cos x}{1 + \cos x}} = L \operatorname{tg} \frac{1}{2} x$$

(Francoeur, 381)

41. $y = \int \cos x\, d\, x$. Sea $\operatorname{sen} x = z \therefore \cos x\, d\, x = d\, z$

$\therefore y = \int d\, z = z = \operatorname{sen} x$. (Roberts, 7)

42. $y = \int \frac{d\, x}{\cos x}\, d\, x = \int \frac{\cos x\, d\, x}{\cos^2 x}$.

Sea

$$z = \operatorname{sen} x \quad \therefore \quad 1 - z^2 = \cos^2 x,$$

$$d\, z = \cos x\, d\, x, \; y = \int \frac{d\, z}{1 - z^2} = L \sqrt{\frac{1 + z}{1 - z}}$$

$$= L \sqrt{\frac{1 + \operatorname{sen} x}{1 - \operatorname{sen} x}} = L \cot\left(\frac{1}{4}\pi - \frac{1}{2} x\right)$$

43. $y = \int \operatorname{sen} x \cos x\, d\, x$. Sea $\operatorname{sen} x = z \therefore \cos x\, d\, x = d\, z$

$$y = \int z\,d\,z = \frac{1}{2}\,\mathrm{sen}^2 x$$

44. $\int \mathrm{sen}^m x \cos^p x . d x$. Sea $p = 2k + 1$, $\mathrm{sen}\,x = u$,

$$\cos x = (1-u^2)^{\frac{1}{2}},\ d x = (1-u^2)^{-\frac{1}{2}} d u$$

$$y = \int u^m (1-u^2)^k\,d u.$$

Sea

$$m=6,\ p=9\ .\ .$$

$$\int \mathrm{sen}^6 x \cos^9 x\,d x = \int u^6 (1-u^2)^4 d u.$$

$$= \frac{1}{7}\sec^7 x - \frac{4}{9}\,\mathrm{sen}^9 x + \frac{6}{11}\,\mathrm{sen}^{11} x - \frac{4}{13}\,\mathrm{sen}^{13} x$$

$$+ \frac{1}{15}\,\mathrm{sen}^{15} x.$$ (Sonnet, 232)

45. $y = \int \sec^3 x\,d x$. Sabemos que la sec. es la inversa del cos:

$$\sec^3 x = \frac{1}{\cos^3 x} = \frac{\cos x}{\cos^4 x},\ \cos x = d\,\mathrm{sen}\,x:$$

$$y = \int \frac{d\,\mathrm{sen}\,x}{(1-\mathrm{sen}^2 x)^2}.$$ Hacemos $\mathrm{sen}\,x = t$ i queda

$$y = \frac{d t}{(1-t^2)^2}$$

Integral que se calculó en el ejercicio 27 del número 53:

$$y = \frac{1}{4}\left[L\frac{1+t}{1-t} + \frac{2t}{1-t^2}\right]$$

Reemplazamos el valor de t = sen x:

$$y = \frac{1}{4}\left[L\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x} + 2 \operatorname{tg} x \sec x\right]$$

Siendo

$$\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x} = \frac{(1+\operatorname{sen} x)^2}{1-\operatorname{sen}^2 x} = \frac{(1+\operatorname{sen} x)^2}{\cos^2 x} = (\sec x + \operatorname{tg} x)^2$$

encontraremos que

$$y = \frac{1}{2}(\sec x + \operatorname{tg} x) + \frac{1}{2}\operatorname{tg} x \sec x.$$

En funcion de la tanjente será:

$$y = \frac{1}{2} L\left(\operatorname{tg} x + \sqrt{\operatorname{tg}^2 x+1}\right) + \frac{1}{2}\operatorname{tg} x\sqrt{\operatorname{tg}^2 x+1}\,)$$

45. $y = \int \frac{dx}{1 + a\cos x + b \operatorname{sen} x}$. Sea $x = y - \operatorname{arc\,tg}\frac{a}{b}$

y sea $\operatorname{arc\,tg}\frac{a}{b} = m$

$$\therefore \operatorname{tg} m = \frac{a}{b},\ \cos m = \frac{b}{\sqrt{a^2+b^2}},\ \operatorname{sen} m = \frac{a}{\sqrt{a^2+b^2}} :$$

$$a \cos x = a \cos y \cos m + a \operatorname{sen} y \operatorname{sen} m$$

$$\underline{b \operatorname{sen} x = b \operatorname{sen} y \cos m - b \operatorname{sen} m \cos y}$$

$$a \cos x + b \operatorname{sen} x = \operatorname{sen} y\,(a \operatorname{sen} m + b \cos m)$$

$$+ \cos y\,(a \cos m - b \operatorname{sen} m)$$

$$= \operatorname{sen} y \left(\frac{a\,b}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}\right)$$

$$+ \cos y \left(\frac{a\,b}{\sqrt{a^2+b^2}} - \frac{a\,b}{\sqrt{a^2+b^2}}\right)$$

$$= \sqrt{a^2+b^2}\,\operatorname{sen} y$$

$$\therefore\ y = \int \frac{d\,y}{1 + \sqrt{a^2+b^2}\,\operatorname{sen} y}\text{, forma ya tratada.}$$

46. $\displaystyle\int \frac{d\,x}{\operatorname{tg} x - \operatorname{tg} a}$ · Sea $\operatorname{tg} x = t \ \therefore\ \sec^2 x\, d\,x = d\,t$,

$$1 + \operatorname{tg}^2 x = 1 + t^2 \ \therefore\ d\,x = \frac{d\,t}{1+t^2}$$

$$y = \int \frac{d\,t}{(1+t^2)\,(t - \operatorname{tg} a)} = \cos^2 a \int \frac{d\,t}{t - \operatorname{tg} a} - \cos^2 a \int \frac{t\,d\,t}{1+t^2}$$

$$- \operatorname{sen} a \cos a \int \frac{d\,t}{1+t^2}$$

$$= \cos^2 a\, L\, \frac{\operatorname{tg} x - \operatorname{tg} a}{\sqrt{1+\operatorname{tg}^2 a}} - x \operatorname{sen} a \cos a$$

(Tannery, 553)

59. *Auxiliar inversa*, o sea sustituir x por $\frac{1}{z}$ o $\frac{1}{x}$ por z.

1. $\int \frac{x\,dx}{(a+b\,x)^3}$ · Sea $a+b\,x=\frac{1}{z}$ ∴ $dx=-\frac{1}{b\,z^2}\,dz$;

uego,

$$x=\frac{1-a\,z}{b\,z},\quad x\,dx=\frac{a\,z-1}{b^2\,z^3}\,dz$$

$$\therefore \int \frac{x\,dx}{(a+b\,x)^3}=\frac{1}{b^2}\int (a\,z-1)\,dz=\frac{1}{b^2}\left(\frac{1}{2}\,a\,z^2-z\right)$$

$$=\frac{a-\frac{2}{z}}{2\,b^2}\,z^2=-\left(\frac{1}{2\,b^2}+\frac{x}{b}\right)\frac{1}{(a+b\,x)^2}$$

(Peacock, 269)

2. $\int \frac{dx}{x\sqrt{x^2-1}}$ · Sea $x=\frac{1}{u}$ ∴ $dx=-\frac{du}{x}$;

luego,

$$\frac{dx}{u}=-\frac{du}{u};$$

queda:

$$\int \frac{dx}{x\sqrt{x^2-1}}=\int \frac{-du}{u\sqrt{\frac{1}{4^2}-1}}=\int \frac{-du}{\sqrt{1-u^2}}=\text{arc cos}\,\frac{1}{x}$$

3. $\int \frac{dx}{x(a+bx^2)}$. Sea $x^2 = \frac{1}{z}$ ∴ $\frac{dx}{x} = -\frac{dz}{2z}$;

luego

$$y = -\frac{1}{2a}\int \frac{a\,dz}{az+b} = -\frac{1}{2a} L(az+b)$$

$$= \frac{1}{a} L \sqrt{\frac{x^2}{a+bx^2}}$$

4. $\int \frac{dx}{x\sqrt{x^2-a^2}}$. Sea $x = \frac{1}{y}$ ∴ $\frac{dx}{x} = -\frac{dy}{y}$

$$= \int - \frac{dy}{y\sqrt{\frac{1}{y^2} - a^2}} = -\int \frac{dy}{\sqrt{1-(ay)^2}} = -\frac{1}{a} \text{arc sen}\, a\, x.$$

(Todhunter, II, 13)

5. $\int \frac{dy}{x\sqrt{a^2 \pm a^2}}$. Sea $x = \frac{1}{y}$

$$= \frac{1}{a} L \frac{x}{a + \sqrt{a^2 \pm x^2}}$$

6. $\int \frac{dx}{x\sqrt{2ax-a^2}}$. Sea $x = \frac{1}{1-z}$ ∴ $\frac{dx}{x} = \frac{a\,dz}{1-z}$

$$\int \frac{a\,dz}{(1-z)\sqrt{\frac{2a^2}{1-z} - a^2}} = \frac{1}{a}\int \frac{dz}{\sqrt{1-z^2}} = \frac{1}{a} \text{arc sen} \frac{x-a}{x}$$

7. $\int \frac{dx}{\sqrt{4x^2-4x-1}}$. Sea $x=\frac{1}{z}$ $\therefore \frac{dx}{x}=-\frac{dz}{z}$

$$\therefore y=\int \frac{-dz}{\sqrt{4-4z-z^2}}=\int \frac{-dz}{\sqrt{8-(z+2)^2}}=\text{arc cos}\,\frac{2x+1}{2\sqrt{2}\,x}$$

(Brahy, 14)

8. $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}}$. Traspongamos la variable:

$$\frac{1}{(1-x^2)^{\frac{3}{2}}}=\frac{1}{\left[x^2\,(x^{-2}-1)\right]^{\frac{3}{2}}}=\frac{x^{-3}}{(x^{-2}-1)^{\frac{3}{2}}}$$

$$=-\frac{1}{2}\,\frac{-2\,x^{-3}}{(x^{-2}-1)^{\frac{2}{8}}}$$

Sea ahora,

$$x^{-2}-1=z \quad \therefore \quad -2\,x^{-3}\,dx=dz$$

$$y=-\frac{1}{2}\int z^{-\frac{3}{2}}\,dz=\frac{x}{\sqrt{1-x^2}}\,.$$

9. $\frac{du}{dx}=\frac{1}{x^m(u+b\,x^n)}$. Sea $x=\frac{1}{z}$ $\therefore$ $\frac{dx}{dz}=-\frac{1}{z^2}$,

$$\therefore \frac{du}{dz}=\frac{du}{dx}\cdot\frac{dx}{dz}=-\frac{1}{z^2}\cdot\frac{du}{dx}=-\frac{z^{m+n-2}}{(az+b)^n}$$

$$\therefore u = -\int \frac{z^{m+n-2}}{z\,(a\,z+b)^n}\,.$$

hacemos ahora

$$a\,z+b=t \;\therefore\; a\,d\,z = d\,t,\; z = \frac{t-b}{a}$$

$$u = -\int \left(\frac{t-b}{a}\right)^{m+n-2} \cdot \frac{d\,t}{a\,t^n}$$

$$= -\frac{1}{a^{m+n-1}} \int \frac{(t-b)^{m+n-2}}{t^n}\,d\,t$$

Desarrollamos por el binomio e integramos cada término.

(Hall, 222)

10. $\int \frac{d\,x}{x\,\sqrt{a^2 \pm x^2}}$. Sea $x = \frac{1}{z}$ i sea $a = \frac{1}{m}$

$$y = -\int \frac{d\,z}{\sqrt{z^2 \pm m^2}} = -L\,\frac{a \pm \sqrt{a^2 \pm x^2}}{a\,x}$$

(Cox, 34)

60. *Auxiliar producto.*

1. $\int \frac{d\,x}{x\,(a+b\,x)^2}$. Sea $a+b\,x = z\,x \therefore d\,x = -\frac{a}{(z-b)^2}\,dz$

$$y = -\frac{1}{a^2}\int \frac{z-b}{z^2}\,d\,z = -\frac{1}{a^2}\left[\,L\,(a+b\,x) + \frac{b}{a+b\,x}\right]$$

2. $\int \frac{dx}{x^m (a+bx)^n}$ Sea $a+bx=zx \therefore x=\frac{a}{z-b}$,

$$dx=-\frac{a}{(z-b)^2}dz,$$

$$y=-\frac{1}{a^{m+n-1}}\int \frac{(z-b)^{m+n-2}}{z^n}dz.$$

Es integrable cuándo sea $m+n>1$

(Williamson, II, 6)

3. $y=\int \sqrt{a^2-x^2}\,dx$ Sea $\sqrt{a^2-x^2}=(a-x)z$

$$dy=\frac{8a^2z^2dz}{(1+z^2)^3}=-\frac{8a^2\ dz}{(1+z^2)^3}+\frac{8a\ dz}{(1+z^2)^2}$$

$$y=\frac{1}{2}x\sqrt{a^2-x^2}+a^2 \operatorname{arc}\left(\operatorname{tg}=\sqrt{\frac{a+x}{a-x}}\right)$$

(Fraucoeur, 367)

4. $\int \frac{dx}{\sqrt{1-x^2}}$. Sea $\sqrt{1-x^2}=(1-x)z$

$$1-x^2=z^2-2xz^2+x^2z^2$$

$$x^2(1+z^2)-2xz^2+z^2-1=0$$

$$x^2-\frac{2z^2}{1+z^2}x+\frac{z^2-1}{z^2+1}=0$$

$$\therefore\ x = \frac{z^2-1}{z^2+1},\ dx = \frac{4\,z\,dz}{(z^2+1)^2}$$

$$\int\frac{dx}{\sqrt{1-x^2}} = 2\int\frac{dz}{1+z^2} = 2\ \text{arc tg}\sqrt{\frac{1+x}{1-x}} = \text{arc sen } x.$$

5. $\int\frac{dx}{\sqrt{a+bx+x^2}}$ Sean α, β las raices reales:

$$\sqrt{a+bx+x^2} = \sqrt{(x-\alpha)(x-\beta)} = (x-a)\,z$$

$$\beta - x = (x-a)\,z^2\ \therefore\ x = \frac{\beta+\alpha z^3}{1+z^2}$$

$$dx = \frac{2(\alpha-\beta)\,z}{(1+z^2)^2}\,dz,\ y = -2\int\frac{dz}{1+z^2}$$

$$= -2\ \text{arc tg}\sqrt{\frac{\beta-x}{x-\alpha}}.$$

6. $\int\frac{dx}{x\ \sqrt{1-x^2}}$ · Sea $\sqrt{1-x^2} = x\,z-1$,

$$\therefore\ 1-x^2 = x^2z^2 - 2\,x\,z + 1 \quad \text{o} \quad x = \frac{2\,z}{1+z^2},$$

$$x\,z = \frac{2\,z^2}{1+z^2},\ \sqrt{1-x^2} = \frac{z^2-1}{z^2+1},\ dx = \frac{2+2\,z^2-4\,z^2}{1+z\,2}$$

$$\therefore\ y = -\int\frac{dz}{z} = L\,\frac{x}{1+\sqrt{1-x^2}}$$

7. $$\int \frac{dx}{x\sqrt{1 \pm x^2}} = L \frac{x}{1 + \sqrt{1 \pm x^2}}$$

8. $$\int \frac{dx}{x\sqrt{a^2 \pm x^2}} = \frac{1}{a} L \frac{x}{a + \sqrt{a^2 \pm x^2}}$$

(Brahy, 17)

9. $$\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}} \quad \text{Sea} \quad \frac{x\sqrt{2}}{1+x^2} = u$$

$$y = \frac{1}{\sqrt{2}} L \frac{\sqrt{1+x^4} + \sqrt{2}}{1-x^4}$$

(Euler, IV, 22)

10. $$\int \frac{\sqrt{1+x^4}}{1-x^4} dx.$$

Sea

$$u = \frac{x\sqrt{2}}{\sqrt{1+x^4}}$$

$$y = \frac{1}{2\sqrt{2}} L \frac{\sqrt{1+x^4} + x\sqrt{2}}{1-x^2} + \frac{1}{2\sqrt{2}} \text{arc sen} \frac{x\sqrt{2}}{1+x^2}$$

61. *Auxiliar potencia.*

1. $y = \int x\sqrt{a-x}\, dx$. Sea $a-x = z^2$ . . $x = a - z^2$,

$$dx = -2z\,dz,\quad y = -\int (a - z^2)^2 z^2\,dz$$

$$= \frac{2}{3}\,a\,(a - x)^{\frac{3}{2}} - \frac{2}{5}\,(a - x)^{\frac{5}{2}}\,.$$

(Roberts, 10)

2. $y = \int x^2 \sqrt{a+x}\,dx.$ Sea $a + x = t^2$ $\therefore$ $dx = 2t\,dt$

$$y = 2\int (t^2 - a)^2 t^2\,dt = 2\left(\frac{t^7}{7} - \frac{2at^5}{5} + \frac{a^2t^3}{3}\right)$$

$$= 2\,(a + x)^{\frac{3}{2}}\left\{\frac{(a+x)^2}{7} - \frac{2a}{5}\,(a + x) + \frac{a^2}{3}\right\}$$

(Todhunter, 18)

3. $y = \int \frac{dx}{\sqrt{1+x^2}}$. Hacemos $1 + x^2 = z^2$

Diferenciamos:

$$2x\,dx = 2z\,dz \;\therefore\; \frac{dx}{z} = \frac{dz}{x}$$

Componemos:

$$\frac{dx + dz}{z+x} = \frac{d\,(x+z)}{z+x} = \frac{dx}{z}$$

$$\therefore\; y = \int \frac{dx}{z} = \frac{d\,(z+x)}{z+x} = L\,(z + x)$$

$$\therefore \int \frac{dx}{\sqrt{1-x^2}} = L\,(x + \sqrt{1+x})$$

4. $\int \frac{x\,dx}{\sqrt{1-x^2}}$ Sea $1-x^2=z^2 \therefore x\,dx = -z\,dz.$

$$y = -\int dz = -z = -\sqrt{1-x^2}$$

5. $\int \frac{dx}{\sqrt{a^2-x^2}}$ Sea $x^2+a=z^2 \therefore \frac{dx}{z} = \frac{dz}{x},$

$$\frac{d\,(x+z)}{x+z} + \frac{dx}{z}$$

$$y = \int \frac{d\,(x+z)}{x+z} = L\,(x + \sqrt{a^2+x^2})$$

(Osborne, 185).

5. $\int \frac{x\,dx}{\sqrt{a^2+x^2}}$ Sea $a^2+x^2=t^2 \therefore x\,dx = t\,dx$

$$y = \sqrt{a^2+x^2}$$

7. $\int \frac{\sqrt{x}+1}{2x\sqrt{x}}\,dx.$ Sea $x=z^2 \therefore dx = 2z\,dz$

$$y = \int \frac{z+1}{z}\,dz = \int (z^{-1}+z^{-2})\,dz$$

$$= L\,z - \frac{1}{z} = L\sqrt{x} - \frac{1}{\sqrt{x}}$$

8. $\int \frac{dx}{\sqrt{x^2 \pm 1}}$. Sea $x^{-2} \pm 1 = u^2$

$$2x\,dx = 2u\,du$$

$$\therefore \frac{d(x+u)}{x+u} = \frac{dx}{\sqrt{x^2 \pm 1}}$$

$$\int \frac{dx}{\sqrt{x^2 \pm 1}} = L\left(x + \sqrt{x^2 \pm 1}\right)$$

(Jarrett, 123)

9. $\int \frac{dx}{x^2\sqrt{1+x^2}}$. Sea $x^{-2} + 1 = t^2$

$$y = -\frac{\sqrt{x^2+1}}{x}$$

10. $$\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = \int \frac{\frac{dx}{dt}}{x^3(a^2x^{-2}+1)^{\frac{3}{2}}}\,dt = -\frac{1}{a^2}\int \frac{dt}{t^2}.$$

$$= \frac{x}{a^2\sqrt{a^2+x^2}}$$

11. $\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$. Sea $x-\alpha = z$

$$\frac{dx}{\sqrt{x-\alpha}} = 2\,dz,$$

$$y = \int \frac{2\,d\,z}{\sqrt{z^2 + \alpha - \beta}}$$

$$= 2 \log \left(\sqrt{x-\alpha} + \sqrt{x-\beta}\right)$$

12. $\int \frac{x\,d\,x}{\sqrt{a^4 - x^4}}$ · Sea $x^4 = a^4 z^2,\ x\,d\,x = \frac{1}{2}\,a^2 d\,z$

$$\therefore\ y = \frac{1}{2}\ \text{arc sen}\ \frac{x^2}{a^2}$$

(Raffy, 84)

62. *Auxiliar binomia.*

1. $y = \int \sqrt{a^2 + x^2}\,d\,x.$ Sea $\sqrt{a^2 + x^2} = z - x$

$$y = \int (z - x)\,d\,x = -\frac{1}{2}\,x^2 + \int z\,d\,x.$$

Para eliminar la d x, elevamos al cuadrado

$$a^2 + x^2 = (z - x)^2 \ \therefore\ x = \frac{z^2 - a^2}{2\,z},$$

$$d\,x = \frac{z^2 + a^2}{2\,z^2}\,d\,z$$

$$y = -\frac{1}{2}\,x^2 + \int \frac{z^2 + a^2}{2\,z}\,d\,z$$

$$= -\frac{1}{2}\,x^2 + \frac{1}{2}\int z\,d\,z + \frac{a^2}{2}\int \frac{d\,z}{z}$$

$$=\frac{1}{2}x\sqrt{a^2+x^2}+\frac{a^2}{2}L(x+\sqrt{a^2+x^2})$$

(Francoeur, 366)

2. $dy=\sqrt{c+bx+x^2}\,dx$. Representemos el radical por $z+x$ i elevemos al cuadrado:

$$bx+c=z^2+2xz \quad \therefore \quad x=\frac{z^2-c}{b-2z} \quad \therefore \quad dx=2\frac{bz-z^2-c}{(b-2z)^2}dz$$

$$y=\int\left(z+\frac{z^2-c}{b-2z}\right).2\frac{bz-z^2-c}{(b-2z)^2}dz$$

$$=2\int\frac{bz-z^2-c)^2}{(b-2z)^3}dz. \quad \text{(Timmermans, 272)}$$

La integracion se puede hacer por division.

3. $\int\frac{dx}{\sqrt{1+x^2}}$ Sea $\sqrt{1+x^2}=v-x$,

$$\therefore v^2-2xv=1$$

$$\int\frac{dx}{\sqrt{1+x^2}}=\int\frac{dx}{v-x}=\int\frac{dv}{v}$$

$$=L(x+\sqrt{1+x^2}) \quad \text{(Pauly, 173)}$$

4. $\int\frac{dx}{\sqrt{x^2+a^2}}$ Sea $\sqrt{x^2+a^2}=z-x \therefore a^2=z^2-2xz$,

$$\frac{d\,x}{\sqrt{x^2+a^2}}=\frac{d\,x}{z-x}=d\,L\,z$$

$$y=L\,(x+\sqrt{x^2+a^2})$$

(Todhunter, II, 9)

5. $\int\frac{d\,x}{\sqrt{a^2+x^2}}$. Sea $\sqrt{a^2+x^2}=x+z\therefore d\,x-\frac{z^2+a^2}{2\,z^2}dz$

$$y=-\int\frac{d\,z}{z}=L\,(x+\sqrt{x^2+a^2})$$

(Laurent, III, 41)

6. $\int\frac{d\,x}{\sqrt{x^2\pm a^2}}=L\,(x+\sqrt{x^2\pm a^2})$

7. $\int\frac{d\,x}{\sqrt{2\,a\,x-x^2}}$. $x=a-z\therefore\frac{dx}{dz}=-1$

$$y=-\int\frac{d\,z}{\sqrt{a^2-z^2}}=\text{arc cos}\,\frac{a-x}{a}.$$

8. $\int\frac{m\,a^m\,x^{m-1}}{\sqrt{(a\,x)^{2\,m}\pm b^{2\,n}}}d\,x=L\,\frac{(a\,x)^m+\sqrt{(a\,x)^{m\,2}+b^{2\,n}}}{b^n}$

(Brahy, 19)

9. $\int\frac{d\,x}{\sqrt{A\,x^2+B\,x+C}}$. Se representa el radical por

$z-x\sqrt{A}$ i se eleva al cuadrado:

$$B\,d\,x = 2\,z\,d\,z - 2\,z\,A\,d\,x - 2\,x\,\sqrt{A}\,d\,z:$$

$$y = \frac{1}{\sqrt{a}}\,L\left(\frac{1}{2}\,B + z\,\sqrt{a}\right) = \frac{1}{\sqrt{a}}\,L\,(A\,x + \frac{B}{2}$$

$$+\sqrt{A}\,\sqrt{A\,x^2 + B\,x + C}\,)$$

(Moigno, II, 20)

63. *Sustitucion trigonométrica.*

1. $\int \sqrt{1-x^2}\,d\,x$. Sea $x = \text{sen}\,\theta$

$$\therefore\ d\,x = \cos\theta\,d\,\theta,\ \cos\theta = \sqrt{1-x^2}$$

$$\therefore\ y = \int \cos^2\theta\,d\,\theta = \int \frac{1+\cos 2\,\theta}{2}\,d\,\theta$$

Pero, de $\text{sen}\,\theta = x$ sale $\theta = \text{arc sen}\,x$; i de $\text{sen}\,2\,\theta = 2\,\text{sen}\,\theta\cos\theta = 2\,x\,\sqrt{1-x^2}$:

$$\int \sqrt{1+x^2}\,d\,x = \frac{1}{2}\,(\text{arc sen}\,x + x\,\sqrt{1-x^2})\ .$$

(Edwards, 38)

2. $\int \sqrt{1+x^2}\,d\,x$. Hacemos $x = \text{tg}\,\theta\ \therefore\ d\,x = \sec^2\theta\,d\,\theta$ $y = \int \sec^3\theta\,d\,\theta$. Esta integral se calculó en el ejercicio 44 del número 58. Reemplazamos ahí $\text{tg}\,\theta$ por x:

$$\int \sqrt{1+x^2}\, d\, x = \frac{1}{2} L\left(x + \sqrt{x^2+1}\right) + \frac{1}{2} x \sqrt{x^2+1}$$

(Todhunter, 11)

3. $\int \sqrt{x^2-1}\, d\, x$. Hacemos $x = \sec \theta$

$\therefore d\, x = \sec \theta \operatorname{tg} \theta\, d\, \theta$, $\operatorname{tg} \theta = \sqrt{x^2-1}$

$$y = \int \sec \theta \operatorname{tg}^2 \theta\, d\, \theta = \int \sec \theta\, d\, \theta + \int \sec^3 \theta\, d\, \theta.$$

Estas dos integrales son conocidas.

$$\int \sqrt{x^2-1}\, d\, x = \frac{1}{2} x \sqrt{x^2-1} - \frac{1}{2} L\left(x + \sqrt{x-1}\right.$$

(Id., 11)

En la integracion por partes encontraremos métodos mas espeditos.

4. $\int \frac{d\, x}{\sqrt{1-x^2}}$ · Sea $x = \operatorname{sen} y \therefore d\, x = \cos y\, d\, y$,

$$\cos y = \sqrt{1-x^2}.$$

$$y = \int \frac{\cos y\, d\, y}{\cos y} = \int d\, y = y = \operatorname{arc\, sen} x.$$

5. $\int \frac{d\, x}{x \sqrt{x^2 - a^2}}$ · Sea $\frac{a}{x} = \cos y$

$$\therefore \frac{\sqrt{x^2-a^2}}{x} = \operatorname{sen} y, \int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{x^2}{a} \cdot \frac{\operatorname{sen} y}{\operatorname{sen} y} \cdot \frac{dy}{x^2}$$

$$= \frac{1}{a}\int dy = \frac{y}{a} = \frac{1}{a} \operatorname{arc\,cos} \frac{a}{x} = \frac{1}{a} \operatorname{arc\,sen} \frac{x}{a}$$

(H. Cox, 35)

6. $\displaystyle\int \frac{dx}{\sqrt{(\alpha-x)\,(x-\beta)}}$ · Sea $x = \alpha \operatorname{sen}^2 \phi + \beta \cos^2 \phi$

$$\therefore \alpha - x = (\alpha-\beta) \cos^2 \phi, \; x - \beta = (\alpha-\beta) \operatorname{sen}^2 \phi, \quad (1)$$

$$(\alpha-\beta)\,(x-\beta) = (\alpha-\beta)^2 \operatorname{sen}^2 \phi \cos^2 \phi$$

$$dx = 2\,(\alpha-\beta) \operatorname{sen} \phi \cos \phi \, d\phi$$

$$y = \int \frac{2\,(\alpha-\beta) \operatorname{sen} \phi \cos \phi \, d\phi}{(\alpha-\beta) \operatorname{sen} \phi \cos \phi} = \int 2\, d\phi = 2\,\phi$$

Dividiendo (1):

$$\frac{\operatorname{sen}^2 \phi}{\cos^2 \phi} = \frac{x-\beta}{\alpha+x} \quad \therefore \quad \operatorname{tg} \phi = \sqrt{\frac{x-\beta}{\alpha-x}}$$

$$\therefore \phi = \operatorname{arc\,tg} \sqrt{\frac{x-\beta}{\alpha-x}}$$

(Roberts, 17)

7. $\displaystyle\int \frac{x^2\, dx}{\sqrt{1-x^2}}$ · Sea $x = \cos z$ $\therefore$ $dx = -\operatorname{sen} z\, dz$

$$y=\int\frac{\cos^2 z-\operatorname{sen} z)\,dz}{\operatorname{sen} z}=-\int\cos^2 z\,d\,z=-\frac{1}{2}z-\frac{1}{4}\operatorname{sen}^2 z.$$

$$\operatorname{sen} 2z=2\operatorname{sen} z\cos z=2\sqrt{1-\cos^2 z}\cos z=2\sqrt{1-x^2}\,x.$$

$$\therefore y=-\frac{1}{4}\operatorname{arc}\cos x-\frac{1}{2}x\sqrt{1-x^2}$$

(J. Bertrand, II, 8)

8. $\int\frac{d\,x}{(x^2+2\,b\,x+c)^n}$. Hacemos $x+b=\sqrt{c-b^2}\ \operatorname{tg} t$

$$d\,x=\sqrt{c-b^2}\sec^2 t\,d\,t$$

$$x^2+2\,b\,x+b^2=(c-b^2)\operatorname{tg}^2 t$$

$$x^2+2\,b\,x=(c-b^2)\operatorname{tg}^2 t-b^2$$

$$x^2+2\,b\,x+c=(c-b^2)\operatorname{tg}^2 t+(c-b^2)$$

$$=(c-b^2)(1+\operatorname{tg}^2 t)$$

$$=(c-b^2)\sec^2 t$$

$$y=\frac{1}{c-b^2}\int\cos^{n-2} t\,d\,t.$$

En las reducciones sucesivas se dará la integral.

(Tannery, 521)

64. *Racionalizacion.*—Operacion, cuyo objeto es hacer racional una espresion irracional. Se emplea para ésto el método de sustitucion.

I. *Funciones monomias.*

1. $y = \int \sqrt{x}\, dx$. Hacemos $x = z^2 \therefore dx = 2\, z\, dz$

$$\therefore y = \int 2\, z^2\, dz = \frac{2}{3} z^2 = \frac{2}{3} x^{\frac{2}{3}}$$

II. *Suma de radicales.* La variable auxiliar tiene por esponente el menor múltiplo comun de los índices.

$$2. \quad \int \left(\sqrt{x} + \sqrt[3]{x} \right) dx = \int \left(x^{\frac{1}{2}} + x^{\frac{1}{3}} \right) dx.$$

El menor múltiplo comun de los índices o denominadores 2 i 3, es 6:

Hacemos

$$x = z^6 \quad \therefore \quad dx = 6\, z^5\, dz.$$

Sustituyamos i obtenemos la funcion de términos racionales:

$$y = \int (z^3 + z^2)\, 6\, z^5\, dz = 6 \int (z^2 + {}^3)\, dz.$$

$$= \frac{2}{3} x^{\frac{3}{2}} + \frac{3}{4} x^{\frac{4}{3}}$$

Este método se emplea solamente en las fracciones polinomias de términos irracionales.

III. *Fracciones de términos irracionales.*

3. $y = \int \frac{\sqrt{x}}{\sqrt{x} - 1} d\, x.$ (Francoeur, II, 364)

Sea

$$x = z^2 \quad \therefore \quad d\,x = 2\,z\,d\,z;$$

queda:

$$y = \int \frac{2\,z\,2}{z^2 - 1}\,d\,z = \int \left(1 + \frac{1}{z^2 - 2}\right) d\,z$$

$$= 2\,x + L\,\frac{\sqrt{x-1}}{\sqrt{x+1}}$$

4. $y = \int \frac{x^{\frac{3}{2}} - x^{\frac{5}{6}}}{x^{\frac{1}{4}}}\,d\,x.$ Efectuamos la division; o

hacemos $x = z^n$, siendo n el menor múltiplo comun de los índices; aquí $n = 12$:

Sea

$$x = z^{12} \quad \therefore \quad d\,x = 12\,z^{11}\,d\,z,\ x^{\frac{3}{2}} = z^{18},$$

$$x^{\frac{5}{6}} = z^{10},\ x^{\frac{1}{4}} = z^3;$$

queda:

$$y = 12 \int \frac{z^{18} - z^{10}}{z^3}\,z^{11}\,d\,z = 12 \int (z^{18} - z^{10})\,z^8\,d\,z$$

$$= 12 \int (z^{27} - z^{18})\, d\, z = 12\, x^{\frac{9}{4}} - x^{\frac{19}{12}}.$$

5. $$\int \frac{1 + \sqrt{x}}{1 + \sqrt[3]{x}} = \int \frac{1 + x^{\frac{1}{2}}}{1 + x^{\frac{2}{3}}}.$$ Hacemos $x = z^6$

$$y = 6 \int \frac{2^2 + z^5}{x^2 + 1}\, d\, z.$$

Efectuamos la division e integramos en seguida:

$$y = \frac{6}{7}\, x^{\frac{7}{6}} - \frac{6}{5}\, x^{\frac{5}{7}} + \frac{3}{2}\, x^{\frac{2}{3}} + 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} - 6\, x^{\frac{1}{6}}$$

$$+ 3\, L\left(x^{\frac{1}{3}} + 1\right) + 6 \text{ arc tg } x^{\frac{1}{6}}.$$

(Serret, II, 24)

6. $$y = \int \frac{x^{\frac{1}{m}} + x^{\frac{2}{n}}}{x^p - x^{\frac{r}{s}}}.$$ Hacemos $x = z^{m \cdot n \cdot s}$.

$$y = \int \frac{z^{n\,s} + z^{2\,m\,s}}{z^{p\,m\,n\,s} - z^{m\,n\,r}}\, d\, z.$$

En seguida se descompone en dos fracciones, se divide o se trasforma en fracciones parciales.

7. $$\int \frac{1 + x^{\frac{1}{4}}}{1 + x^{\frac{1}{3}}}\, d\, x = \frac{1}{11}\, x^{\frac{11}{12}} + \frac{1}{8}\, x^{\frac{3}{2}} - \frac{1}{7}\, x^{\frac{7}{12}} - \frac{1}{4}\, x^{\frac{1}{5}}$$

$$+\frac{1}{3}x^{\frac{1}{4}}+\frac{1}{2^{\frac{5}{2}}}\left[L\ \frac{x^{\frac{1}{6}}-2^{\frac{1}{2}}x^{\frac{1}{12}}+1}{x^{\frac{1}{6}}+2^{\frac{1}{2}}x^{\frac{1}{12}}+1}\right.$$

$$\left.+2\ \text{arc tg}\ \frac{2^{\frac{1}{2}}x^{\frac{1}{12}}}{1-x^{\frac{1}{6}}}\right]$$

(Grégory, 267)

8. $$dy=\frac{1+\sqrt{x}+\sqrt[3]{x^2}}{1+\sqrt[3]{x}}dx.\ \text{Sea}\ x=t^6\ \therefore\ dx=6t^5dt.$$

Sustitúyese i resulta la fraccion de términos racionales:

$$dy=\frac{1+t^3-t^4}{1+t^2}\cdot 6\,t^5\,d\,t.$$

Efectuamos la division:

$$y=6\int\left(-t^7+t^6+t^5-t^4+t^2-1+\frac{1}{1+t^2}\right)d\cdot t$$

$$=-\frac{3}{4}t^8+\frac{6}{7}t+t^6-\frac{6}{5}t^5+2t^3-6t+6\ \text{arc tg}\ t.$$

(Sturm, 334)

9. $$dy=\frac{\sqrt[3]{x}-\sqrt[6]{x}+1}{\sqrt[6]{x}-1}.\ \text{Sea}\ x=z^6$$

$$y=6\left[\frac{1}{7}x^{\frac{7}{6}}+\frac{1}{5}x^{\frac{5}{6}}+\frac{1}{4}x^{\frac{2}{3}}+\frac{1}{3}x^{\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{3}}\right.$$

$$\left.+x^{\frac{1}{6}}+L\left(x^{\frac{1}{6}}-1\right)\right]$$

(Brahy, 21)

10. $dy=\dfrac{\sqrt{x}-1}{6\left(\sqrt[3]{x}+1\right)}dx$. Sea $x=z^6$

$$y=\frac{1}{7}x^{\frac{7}{6}}-\frac{1}{5}x^{\frac{5}{6}}-\frac{1}{4}x^{\frac{2}{3}}+\frac{1}{3}x^{\frac{1}{2}}+\frac{1}{2}x^{\frac{1}{3}}-x^{\frac{1}{6}}$$

$$+\operatorname{arc\,tg} x^{\frac{1}{6}}-L\sqrt{x^{\frac{1}{3}}+1}\,.$$

11. $dy=\dfrac{x^2+\sqrt[3]{(ax+b)^2}}{x+\sqrt{ax+b}}dx$. Sea $ax+b=z^6$

$$y=\frac{6}{a^2}\int\frac{z^5\left[(z^6-b)^2+a^2b^4\right]}{z^6-b+az^3}dx\,.$$

Se integra por division.

12. $dy-\dfrac{\sqrt[3]{x}+x\sqrt{x}+x^2}{x+\sqrt{x}}dx$. Sea $x=z^6$

$$y=\int\left(z^{11}+z-\frac{z}{z^3+1}\right)dz\,.$$

Cada término es de fácil integracion.

13. $$d\,y = \frac{a\sqrt[3]{x^2} - b\sqrt[4]{x^5}}{a\sqrt{x^3} + \sqrt[3]{x^2} + e\,x^2}\,d\,x. \quad \text{Sea } x = z^{12}$$

$$y = 12\int \frac{a - b\,z^7}{c\,z^7 + e\,z^5}\,d\,z$$

Se integra por el método de las fracciones racionales.

(Timmermans, 271)

IV. *Funciones irracionales.*

14. $$d\,y = \left(\frac{a\,x + b}{a^1 x + b^1}\right)^{\frac{m}{n}} d\,x. \quad \text{Sea } \frac{a\,x + b}{a^1 x + b^1} = z^n$$

$$\therefore\; d\,x = \frac{(b^1 - a\,b^1)\,n\,z^{n-1}}{(b - b^1 z^n)^2}$$

$$y = n\,(b\,a^1 - a\,b^1)\int \frac{z^{m+n-1}}{(b - b^1 z^n)^2}\,d\,z$$

Esta funcion racional es de fácil integracion.

15. $\int x^m (a + b\,x^n)^{\frac{p}{q}}$ · Esta forma lleva el nombre de *diferencial binomia* i tiene una gran importancia.

Se puede racionalizar cuando sea entero $\frac{m+1}{n}$: en tal caso se hace $a + b\,x^n = z^q$; i si $\frac{m+1}{n} + \frac{p}{q}$ es entero, se hace $a + b\,x^n = x^n z^q$.

La siguiente aplicacion es de Grégory, páj. 265.

16. $$\int \frac{x}{(a+b\,x)^{\frac{1}{2}}}\,d\,x = \int x\,(a+b\,x)^{-\frac{1}{2}}\,d\,x$$

Aquí $m=1$, $n=1$, $\frac{p}{q}=-\frac{1}{2}$ $\quad\therefore\ \frac{m+1}{n}=2$, hacemos $a+b\,x=z^2$

$$d\,x=\frac{2\,z}{b}\,d\,z,\ x=\frac{b^2-a}{b},$$

$$y=\frac{2}{b^2}\int(z^2-a)\,d\,z=\frac{2}{b^2}\left(\frac{1}{3}\,z^3-a\,z\right)$$

$$=\frac{2}{3\,b^2}(b\,x-2\,a)\,\sqrt{a+b\,x}$$

17. $$\int x^3(a+x)^{\frac{2}{3}}$$

$m=3$, $n=1$, $\frac{p}{q}=\frac{2}{3}$ $\quad\therefore\ \frac{m+1}{n}=4$, $q=3$ $\quad\therefore\ a+x=z^3,\ d\,x=3\,z^2\,d\,z$

$$y=\int(z^3-a)^3z^2d\,z=\int(z^{11}-3\,a\,z^8+3\,a^2z^5-a^3z^2)\,d\,z$$

$$=(a+x)^{\frac{5}{3}}\left[\frac{1}{16}(a+x)^{\frac{11}{3}}-\frac{a}{13}(a+x)^{\frac{8}{3}}+\frac{3a^2}{10}(a+x)^{\frac{5}{3}}\right.$$

$$-\frac{a^3}{7}(a+x)^{\frac{2}{3}}\Big] \cdot$$

18. $$\int \frac{dx}{x^4(1+x^2)^{\frac{1}{2}}} = \int x^{-4}(1+x^2)^{-\frac{1}{2}}dx$$

$$\left.\begin{array}{l} m=-4 \\ n=2 \\ \frac{p}{q}=-\frac{1}{2} \end{array}\right| \therefore \frac{m+1}{n}+\frac{p}{q}=-2 \therefore 1+x^2=x^2z^2$$

$$dx=\frac{xz}{1-z^2}dz, \quad x^4=\frac{1}{(z^2-1)^2}$$

$$y=\int \frac{\frac{xz\,dz}{1-z^2}}{\frac{xz}{(z^2-1)^2}} = \int (1-z^2)\,dz = z-\frac{1}{3}z^3$$

$$=\sqrt{1+x^2}\left(\frac{1}{x}-1-3x\right)$$

19. $$w=\int \frac{dx}{(1+nx^2)\sqrt{1-x^2}} \cdot \quad \text{Sea } t=\frac{x}{\sqrt{1-x^2}}$$

$$dx=\frac{dt}{(1+t^2)^{\frac{3}{2}}} \quad w=\int \frac{\frac{dt}{(1+t^2)^{\frac{3}{2}}}}{\left(1+\frac{nt^2}{1+t^2}\right)\left(\frac{1}{1+t^2}\right)^{\frac{1}{2}}}$$

Simplificando, se obtiene:

$$w = \int \frac{d\,t}{1+(1+n)\,t^2} = \frac{1}{\sqrt{1+n}} \text{ arc tg } \sqrt{1+n}\,t$$

$$= \frac{1}{\sqrt{1+n}} \text{ arc tg } \frac{x}{\sqrt{1-x^2}}$$

(Serret, 49)

20. $\int \sqrt{a+b\,x+x^2}$. Sea $\sqrt{a+b\,x+x^2} = z - x$

Elevando al cuadrado i diferenciando resulta:

$$y = 2 \int \frac{(a+b\,z+z^2)^2}{(b+x\,z)^3}\, d\,z.$$

65. *Conversion de las funciones.*—Bajo este nombre designamos la trasformacion que convierte una funcion irracional en racional o una trascendente en aljebraica.

En los dos números anteriores se dieron numerosos ejemplos de esta clase de trasformacion.

Agregaremos otros mas, a fin de que el principiante se familiarice con estos cambios de formas de las funciones.

1. $d\,y = L^n x \frac{d\,x}{x}$ · Sea $L\,x = z$ ∴ $\frac{d\,x}{x} = d\,z$

La funcion logarítmica se convierte en la racional

$$y = \int z^n d\,x = \frac{1}{n+1} L^{n+1} x.$$

2. $\int \operatorname{sen} x\, d\, x$. Sea $\operatorname{tg} \frac{1}{2} = z = \sqrt{\frac{1-\cos x}{1+\cos x}}$.

Los dos primeros miembros dan:

$$a = 2 \operatorname{arc} \operatorname{tg} z \quad \therefore \quad d\, x = \frac{2\, d\, z}{1+z^2};$$

i los dos últimos,

$$\cos x = \frac{1-z^2}{1+z^2}, \quad \operatorname{sen} x = \frac{2\, z}{1+z^2}.$$

Sustituyendo:

$$\int \operatorname{sen} x\, d\, x = \int \frac{2\, z}{1+z^2} \cdot \frac{2\, d\, z}{1+z^2} = 2 \int (1+z^2)^2\, d\,(1+z^2)$$

$$= -\frac{2}{1+z^2} = -1 - \cos x.$$

Difiere esta integral en —1 de la integral conocida. La auxiliar anterior se emplea en ejemplos mas complicados, como los que siguen:

3. $\int \frac{d\,\theta}{a+b\cos\theta}$. Sea $\cos\theta = \frac{1-z^2}{1+z^2} \therefore d\,\theta = \frac{2\, d\, z}{1+z^2}$.

$$= \int \frac{\frac{2\, d\, z}{1+z^2}}{a + \frac{b(1-z^2)}{1+z^2}} = \frac{d\, z}{a+b} \int \frac{d\, z}{1 + \frac{a-b}{a+b} z^2}$$

si a > b, tendremos:

$$y = \frac{2}{\sqrt{(a+b)(a-b)}} \text{ arc tg } \sqrt{\frac{a-b}{a+b}}\, z$$

$$= \frac{2}{\sqrt{a^2-b^2}} \text{ arc cos } \frac{a \cos\theta + b}{a + b \cos\theta}$$

si a < b:

$$y = \frac{1}{\sqrt{b^2-a^2}} L \frac{\operatorname{tg}\frac{1}{2}\theta + \sqrt{\frac{b+a}{b-a}}}{\operatorname{tg}\frac{1}{2}\theta - \sqrt{\frac{b+a}{b-a}}}$$

si a=b:

$$y = \frac{1}{a} \operatorname{tg} \frac{1}{2}\theta$$

(Sturm, 364)

4. $\int \frac{dx}{a \operatorname{sen} x + b \cos x + c}$ · El método anterior nos da:

$$y = \int \frac{\frac{2\,dz}{1+z^2}}{a \cdot \frac{2z}{1+z^2} + \frac{b(1-z^2)}{1+z^2} + \frac{c(1+z^2)}{1+z^2}}$$

$$= \int \frac{2\,dz}{(c-b)z^2 + 2az + b + c}$$

Esta forma es conocida.

5. $y = \int \mathrm{sen}^m x \cos^n x\, dx$. Sea $z = \mathrm{sen}^2 x$

$\therefore \cos^2 x = 1 - z,\ dz = 2\, \mathrm{sen}\, x \cos x\, dx$

$$y = \pm \frac{1}{2} \int z^{\frac{m-1}{2}} (1-z)^{\frac{n-1}{2}} dz$$

6. $y = \int \mathrm{sen}^4 x \cos x^3 dx$. Segun lo anterior:

$$y = \int z^4 (1 - z^2) = \frac{1}{5} \mathrm{sen}^5 x - \frac{1}{7} \mathrm{sen}^7 x.$$

(Francoeur, 378)

7. $y = \int f(\mathrm{sen}\, x, \cos x)\, dx$. Sen $x = z$

$$\therefore \cos x = \sqrt{1 - z^2},\ dx = \frac{dz}{\sqrt{1 - z^2}}$$

Se obtiene la funcion aljebraica:

$$y = \int f(z, \sqrt{1 - z^2}) \frac{dz}{\sqrt{1 - z^2}}$$

(Moigno, II, 33)

7. $y = \int \mathrm{arc\ sen}\, x\, dx$. Convertimos esta funcion circular en trigonométrica, haciendo $\mathrm{arc\ sen}\, x = z \therefore dz = \frac{dx}{\sqrt{1 - x^2}}$.

$$y = \int z \cos z\, dz.$$

66. *Empleo de las imajinarias.*—1. Para construir una función circular en logarítmica, hacemos $a = 1$, $x = i z$, $i^2 = -1$, $d x = i d z$, en la integral

$$\int \frac{d x}{a^2 - x^2} = \frac{1}{a} L \sqrt{\frac{a+x}{a-x}};$$

se obtiene:

$$\int \frac{i d z}{1+z^2} = \frac{1}{2} L \frac{1+i z}{1-i z}$$

o bien

$$i \operatorname{arc} \operatorname{tg} z = \frac{1}{2} L \frac{1+i z}{1-i z}$$

(Roberts, 14)

2. $\int e^{ax} \cos b x \, d x$. Empleemos las ecuaciones de Euler:

$$\operatorname{sen} x = \frac{e^{ix} - e^{-ix}}{2 i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2};$$

$$y = \int e^{ax} . \frac{e^{bix} + e^{-bix}}{2} dx = \int \frac{e^{(bi+a)x} + e^{(a-bi)x}}{2} dx$$

$$= \frac{1}{2}\left[\frac{e^{(a+bi)x}}{a+bi} + \frac{e^{(a-bi)x}}{a-bi}\right]$$

$$= \frac{a \cos b x + b \operatorname{sen} b x}{a^2 + b^2} e^{ax}$$

3. $$\int e^{ax} \operatorname{sen} bx\, dx = \frac{a \operatorname{sen} bx - b \cos bx}{a^2 + b^2} e^{ax}$$

(Tannery, II, 510)

4. $$dy = \frac{-dx}{\sqrt{1 - x^2}} = \frac{-dx}{i\sqrt{x^2 - 1}}$$

$$\therefore\; yi = L\,(x + \sqrt{x^2 - 1})$$

Sea

$$x = \cos y \;\therefore\; x^2 - 1 = \cos^2 x - 1 = -\operatorname{sen}^2 x;$$

queda

$$\pm\, y i = L\,(\cos y \pm i \operatorname{sen} y).$$

Para demostrar que este resultado es cierto, acudimos a las dos ecuaciones de Euler:

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

$$i \operatorname{sen} y = \frac{e^{iy} - e^{-iy}}{2}$$

Si sumamos miembro a miembro, se obtiene:

$$\cos y + i \operatorname{sen} y = e^{iy};$$

Aplicamos logaritmos:

$$i\, y = L\,(\cos y + i \operatorname{sen} y).$$

CAPITULO VI

INTEGRACION POR PARTES

67. *Idea jeneral del método.*—En el número 34 dimos a conocer la fórmula empleada:

$$\int u\, d\, v = u\, v - \int v\, d\, u,$$

que comprende la sustitucion de u i v como funciones de x, i la descomposicion de un producto en una diferencia.

Este método se emplea en las funciones irracionales i trascendentes.

Para el objeto de comprender todos sus recursos e importancia, hemos considerado diversos casos.

El mas sencillo es cuando la *nueva integral* $\int v\, d\, u$ es de integracion inmediata.

68. *La nueva integral es de integracion inmediata.*

1. $y = \int L\, x\, d\, x.$

Sea

$$u = L\, x \quad \therefore \quad d\, u = \frac{d\, x}{x},$$

$$d\, v = d\, x \quad \therefore \quad v = x.$$

Aquí se hizo la diferenciacion de $u = L\, x$ i la integracion de $d\, v = d\, x$.

Sustituyamos en la fórmula:

$$\begin{cases} \int u\,d\,v = u\,v - \int v\,d\,u; \\ \int L\,x\,d\,x = x\,L\,x - \int d\,x \end{cases}$$

La nueva integral $\int d\,x$ es de integracion inmediata:

$$\int L\,x\,d\,x = x\,L\,x - x.$$

2. $y = \int \text{arc sen } x\,d\,x.$

Sea

$$u = \text{arc sen } x \quad \therefore \quad d\,u = \frac{d\,x}{\sqrt{1-x^2}}$$

$$d\,v = d\,x \quad \therefore \quad v = x$$

$$y = \int \text{arc sen } x\,d\,x = x \text{ arc sen } x - \int \frac{x\,d\,x}{\sqrt{1-x^2}}$$

Aquí

$$\int \frac{x\,d\,x}{\sqrt{1-x^2}} = -\frac{1}{2}\int (1-x^2)^{-\frac{1}{2}} d\,(1-x^2) = -\sqrt{1-x^2}:$$

$$\therefore \int \text{arc sen } x\,d\,x = x \text{ arc sen } x + \sqrt{1-x^2}$$

3. $\int \text{arc cos } x\,d\,x.$

$$\left| \begin{array}{l} u = \text{arc cos } x \quad \therefore \quad d\,u = \dfrac{-d\,x}{\sqrt{1-x^2}} \\ d\,v = d\,x \quad \therefore \quad v = x \end{array} \right.$$

$$y = x \text{ arc} \cos x + \int \frac{x\, d\, x}{\sqrt{1-x^2}} = x \text{ arc} \cos x - \sqrt{1-x^2}$$

4. $\int \text{arc tg } x\, d\, x.$ $\left| \begin{array}{l} u = \text{arc tg } x \quad \therefore \quad d\, u = \dfrac{d\, x}{x + x^2} \\ d\, x = d\, x \quad \therefore \quad v = x \end{array} \right.$

$$y = x \text{ arc tg } x\, d\, x - \int \frac{x\, d\, x}{1+x^2} = x \text{ arc tg } x - \frac{1}{2} \text{ L } (1+x^2)$$

(Humbert, I, 429)

5. $\int \text{arc cot } x\, d\, x = x \text{ arc cot } x + \text{L } (1+x^2)^{\frac{1}{2}}$

6. $\int \text{arc sec } x\, d\, x = x \text{ arc sen } x - \text{ L } (x + \sqrt{x^2-1})$

7. $\int \text{arc cosec } x\, d\, x = x \text{ arc cosec } x + \text{L } (x + \sqrt{x^2-1})$

8. $\int \text{arc vers } x\, d\, x = x \text{ arc vers } x - \int \frac{x\, d\, x}{\sqrt{2\, x - x^2}}$

$$\int \frac{-x\, d\, x}{\sqrt{2\, x - x^2}} = \int \frac{-x+1-1}{\sqrt{2\, x - x^2}}\, d\, x = \sqrt{2x - x^2} - \text{arc vers } x$$

$$\therefore \int \text{arc vers } x\, d\, x = (x-1) \text{ arc vers } x + \sqrt{2\, x - x^2}$$

(Brahy, 33)

9. $\int (1+\text{tg}^2 x)\, d\, x = \int d\, x + \int \text{tg}^2 x\, d\, x.$

$$\text{tg}^2\, x = \frac{\text{sen}^2\, x}{\cos^2\, x} = -\frac{\text{sen } x\, d \cos x}{\cos^2\, x}:$$

Sea

$$u = \text{sen } x \quad \therefore \quad d\,u = \cos x\, d\,x,$$

$$d\,v = \frac{d \cos x}{\cos^2 x} \quad \therefore \quad v = \frac{1}{\cos x};$$

sustituimos:

$$\int_x \text{tg}^2 x = \frac{\text{sen } x}{\cos x} - \int d\,x = \text{tg } x - x;$$

$$\therefore \int (1 + \text{tg}^2 x) = x + \text{tg } x - x = \text{tg } x.$$

10. $\int_x (1 + \cot^2 x) = \int d\,x + \int \cot^2 x\, d\,x$, o bien,

$$= -\int \left[1 + \text{tg}^2 \left(\frac{\pi}{2} - x \right) \right] d \left(\frac{\pi}{2} - x \right) = -\text{tg} \left(\frac{\pi}{2} - x \right)$$

$$= -\cot x.$$

11. $\int x\, e^x\, d\,x.$ Sea $u = x \quad \therefore \quad d\,u = d\,x$

$$d\,v = e^x\, d\,x \quad \therefore \quad v = e^x$$

$$y = x\, e^x - \int e^x\, d\,u = x\, e^x - e^x$$

12. $\int x\, L\, x\, d\,x.$ | $u = L\, x \quad \therefore \quad d\,u = \dfrac{d\,x}{x}$

$d\,v = x\, d\,x \quad \therefore \quad v = \dfrac{1}{2} x^2$

$$y = \frac{1}{2} x^2 L x - \int \frac{1}{2} x^2 . \frac{d x}{x} = \frac{1}{2} x^2 L x - \frac{1}{4} x^2$$

(Appell, 51)

13. $\int x^3 L x d x . \left| \begin{array}{ll} u = L x & \therefore d u = \frac{d x}{x} \\ d v = x^3 d x & v = \frac{1}{4} x^4 \end{array} \right.$

$$y = \frac{1}{4} x^4 L x - \int \frac{1}{4} x^4 . \frac{d x}{x} = \frac{1}{4} x^4 L x - \frac{1}{16} x^4$$

(Bertrand, 9)

14. $\int L x . x^n d x . \left| \begin{array}{ll} u = L x & \therefore d u = \frac{d x}{x} \\ d v = x^n d x & \therefore v = \frac{x^{n+1}}{n+1} \end{array} \right.$

$$y = \frac{x^{n+1}}{n+1} L x - \frac{1}{n+1} \int x^n d x$$

$$= \frac{x^{n+1}}{n+1} \left(L x - \frac{1}{n+1} \right)$$

(Briot, II, 504)

15. $\int x \operatorname{sen} x d x . \left| \begin{array}{ll} u = x & \therefore d u = d x \\ d v = \operatorname{sen} x d x & \therefore v = - \cos x \end{array} \right.$

$$y = - x \cos x + \int d \operatorname{sen} x = - x \cos x + \operatorname{sen} x .$$

(Moigno, II, 15)

16. $\int x \cos x \, d x = x \operatorname{sen} x - \int \operatorname{sen} x \, d x$

$$= x \operatorname{sen} x + \cos x$$

(Duhamel, I, 429)

17. $\int x \cos a x \, d x = x \int \frac{\operatorname{sen} a x}{a} - \frac{1}{a} \int \operatorname{sen} a x \, d x$

$$= \frac{x \operatorname{sen} a x}{a} + \frac{\cos a x}{a^2}$$

18. $\int \cos x \, L \operatorname{sen} x \, d x.$ $\left| \begin{array}{l} u = L \operatorname{sen} x \quad \therefore \quad d u = \cot x \\ d v = \cos x \, d x \quad \therefore \quad v = -\operatorname{sen} x \end{array} \right.$

$$y = -\operatorname{sen} x \, L \operatorname{sen} x - \operatorname{sen} x.$$

19. $\int \operatorname{sen} x \, L \cos x \, d x = \cos x \, L \cos x + \cos x.$

20. $\int x . \operatorname{arc} \operatorname{sen} x \, d x.$ $\left| \begin{array}{l} u = \operatorname{arc} \operatorname{sen} x \quad \therefore \quad d u = \frac{d x}{\sqrt{1 - x^2}} \\ v d = x \, d x \quad \therefore \quad v = \frac{1}{2} x^2 \end{array} \right.$

$$y = \frac{1}{2} x^2 \operatorname{arc} \operatorname{sen} x - \frac{1}{4} \int \frac{x^2 \, d x}{\sqrt{1 - x^2}}$$

$$= \frac{1}{2} x^2 \operatorname{arc} \operatorname{sen} x + \frac{1}{4} \operatorname{arc} \cos x - \frac{1}{4} x \sqrt{1 - z^2}$$

(Bertrand. 10)

21. $\int \frac{x \text{ arc sen } x}{\sqrt{1-x^2}} dx$ | $u = \text{arc sen } x \quad \therefore \quad du = \frac{dx}{\sqrt{1-x^2}}$; $dv = \frac{x\,dx}{\sqrt{1-x^2}} \quad \therefore \quad v = -\sqrt{1-x^2}$

$$y = -\sqrt{1-x^2} \text{ arc sen } x + x.$$

22. $\int f(x) \text{ arc sen } x\, dx = \text{arc sen } x \int f(x)\, dx)$

$$-\int\left(\frac{dx}{\sqrt{1-x^2}}\int f(x)\, dx\right)$$

La integracion por partes trasforma una funcion trascendente en aljebraica. (Francoeur, 378)

23. $\int \frac{x(3-x^2) \text{ arc tg } x}{(1-x^2)^{\frac{3}{2}}} dx.$

$$u = \text{arc tg } x \quad \therefore \quad du = \frac{dx}{1+x^2}$$

$$dv = \frac{x(3-x^2)}{(1+x^2)^{\frac{3}{2}}} dx = \frac{x(2+1-x^2)}{(1-x^2)^{\frac{3}{2}}} dx$$

$$= -(1-x^2)^{-\frac{3}{2}} d(1-x^2) - \frac{1}{2}(1-x\ )^{-\frac{1}{2}} d(1-x^2)$$

$$= \frac{1+x^2}{\sqrt{1-x^2}} dx$$

$$\therefore\ y = \frac{1+x^2}{\sqrt{1-x^2}} \text{ arc tg } x - \int \frac{1+x^2}{\sqrt{1-x^2}} \cdot \frac{dx}{1+x^2}$$

$$= \frac{1+x^2}{\sqrt{1-x^2}} \text{ arc tg } x - \text{arc sen } x.$$

24. $$\int \frac{a-2x}{\sqrt{ax-x^2}} \text{ arc sen} \sqrt{\frac{a-x}{a+x}} \cdot dx$$

Sea

$$u = \text{arc sen} \sqrt{\frac{a-x}{a+x}}.$$

Para diferenciar, hacemos

$$z = \sqrt{\frac{a-x}{a+x}}$$

$$\therefore\ du = d \text{ arc sen } y = \frac{dz}{\sqrt{1-z^2}}$$

$$dz = -\frac{a}{(a-x)^{\frac{1}{2}}(a+x)^{\frac{3}{2}}},\quad 1-z^2 = \frac{2x}{a+x}$$

$$\therefore\ du = -\frac{a}{\sqrt{3}\,(ax-x^2)^{\frac{1}{2}}(a+x)};$$

sea

$$dv = \frac{a-2x}{\sqrt{ax-x^2}}\,dx \quad \therefore \quad v = \sqrt{ax-x^2}$$

$$y = \sqrt{a\,x - x^2}\ \text{arc sen}\ \frac{\sqrt{a-x}}{\sqrt{a+x}} + \frac{a}{\sqrt{2}}\int \frac{a}{a+x}$$

La nueva integracion ès de inmediata integracion.

25. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\ \text{arc sec}\ \sqrt{\frac{1+x^2}{1-x^2}}$

Sea

$$u = \text{arc sec}\ \sqrt{\frac{1+x^2}{1-x^2}},\quad z = \sqrt{\frac{1+x^2}{1-x^2}}$$

$$\therefore\ d\,u = d\ \text{arc sec}\ z = \frac{d\,z}{z\sqrt{z^2-1}}:$$

$$L\,z = \frac{1}{2}\left[L(1+x^2) - L((1-x^2)\right].$$

$$\frac{d\,z}{z} = \frac{1}{2}\left[\frac{2x}{1+x^2} + \frac{2x}{1-x^2}\right)d\,x$$

$$d\,z = \frac{x\,.\,d\,x}{\sqrt{1+x^2}\,.\,\sqrt{(1-x^2)^3}}$$

$$d\,u = \frac{\dfrac{x}{\sqrt{1+x^2}\,\sqrt{(1-x^2)^3}}}{\dfrac{\sqrt{1+x^2}}{\sqrt{1-x^2}}\,.\,\sqrt{\dfrac{1+x^2}{1-x^2}-1}}$$

$$= \frac{x(1-x^2)}{x(1+x^2)(1-x^2)^{\frac{3}{2}}} = \frac{1}{(1+x^2)(1+x^2.)^{\frac{1}{2}}}$$

Sea

$$d\,v = \frac{x\,d\,x}{\sqrt{1-x^2}} \quad .\cdot. \quad v = -\sqrt{1-x^2}$$

$$y = -\sqrt{1-x^2}\ \text{arc sec}\ \sqrt{\frac{1+x^2}{1-x^2}} + \text{arc tg}\ x.$$

(Brahy, 34)

69. *Trasposicion de la nueva integral.*—Al aplicar la fórmula de la integracion por partes,

$$\int u\,d\,v = u\,v - \int v\,d\,u,$$

puede suceder que la nueva integral $\int v\,d\,u$ sea igual a la integral primitiva $\int u\,d\,v$, o múltiplo de ella; en tal caso tendremos:

$$\int u\,d\,v = u\,v - k\int u\,d\,v.$$

Trasponemos la nueva integral:

$$(1+k)\int u\,d\,v = u\,v$$

$$.\cdot.\quad \int u\,d\,v = \frac{1}{1+k}\,u\,v.$$

De esta manera queda eliminada la nueva integral.

1. $\int x^n d\,x$. Sea $\left|\begin{array}{ll} u = x^n & .\cdot.\ d\,u = n\,x^{n-1}\,d\,x \\ d\,v = d\,x & .\cdot.\ \ v = x \end{array}\right.$

$$\therefore \int x^n dx = x^n . x - \int n x . x^{n-1} dx$$

$$= x^{n+1} - n \int x^n dx .$$

Trasponiendo la nueva integral, se obtiene:

$$n \int x^n dx + \int x^n dx = x^{n+1}$$

ó

$$(n+1) \int x^n dx = x^{n+1}$$

$$\therefore \int x^n dx = \frac{x^{n+1}}{n+1}$$

(Montférrier, II, 172)

2. $\int \sqrt{1+x^2}\, dx$. Descompongamos el radical:

$$\int \sqrt{1+x^2}\, dx = \int \frac{dx}{\sqrt{1+x^2}} + \int \frac{x^2}{\sqrt{1+x^2}}\, dx.$$

La primera integral es conocida:

$$\int \frac{dx}{\sqrt{1+x^2}} = L \left(x + \sqrt{1+x^2}\right)$$

La segunda se integra por partes:

Sea

$$u = x \qquad \therefore \qquad du = dx$$

i sea

$$d\,x = \frac{x}{\sqrt{1+x^2}}\,d\,x \qquad \therefore \; v = \frac{1}{2}\int (1+x^2)^{\frac{1}{2}}\,d(1+x^2)$$

$$= \sqrt{1+x^2}$$

$$\therefore \int x \cdot \frac{x}{\sqrt{1+x^2}}\,d\,x = x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,d\,x.$$

Sustituyendo:

$$\int \sqrt{1+x^2}\,d\,x = L\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2} - \int \sqrt{1+x^2}\,d\,x$$

Trasponiendo la nueva integral i reduciendo, se obtiene:

$$\int (1+x^2)\,d\,x = \frac{1}{2}\left[L\left(x+\sqrt{1+x^2}\right) + x\sqrt{1+x^2}\right]$$

Serret, 215)

3. $\int \sqrt{x^2+a}\,d\,x$. Sea $u = \sqrt{x^2+a} \; \therefore \; d\,u = \frac{x\,d\,x}{\sqrt{x^2+a}}$

$$d\,v = d\,x \qquad \therefore \qquad v = x.$$

$$\int \sqrt{x^2+a}\,d\,x = x\sqrt{x^2+a} - \int \frac{x^2}{\sqrt{x^2+a}}\,d\,x$$

Pero

$$\frac{x^2}{\sqrt{x^2+a}} = \frac{x^2+a-a}{\sqrt{x^2+a}} = \frac{x^2+a}{\sqrt{x^2+a}} - \frac{a}{\sqrt{x^2+a}}$$

$$= \sqrt{x^2 + a} - \frac{a}{\sqrt{x^2 + a}} :$$

$$\int \sqrt{x^2 + a}\, dx = x \sqrt{x^2 + a} - \int \sqrt{x^2 + a} + \int \frac{a}{\sqrt{x^2 + a}}$$

Trasponiendo i reduciendo:

$$\int \sqrt{x^2 + a}\, dx = \frac{1}{2}\left(x \sqrt{x^2 + a} + \int \frac{a}{\sqrt{x^2 + a}}\, dx \right)$$

$$= \frac{1}{2}\left[L\left(x + \sqrt{a + x^2}\right) + x \sqrt{x^2 + a} \right]$$

(Tannery, 523)

4. $$\int \sqrt{x^2 - 1}\, dx = \int \frac{x^2}{\sqrt{x^2 - 1}}\, dx - \int \frac{1}{\sqrt{x^2 - 1}}\, dx \quad (1)$$

La primera integral se integra por partes.

Sea

$$u = x,\ dv = \frac{x\, dx}{\sqrt{x^2 - b}}$$

$$\therefore\ du = dx, \quad v = \sqrt{x^2 - 1}$$

Sustituyendo:

$$\int \frac{x^2}{\sqrt{x^2 - 1}}\, dx = x \sqrt{x^2 - 1} - \int \sqrt{x^2 - 1} \qquad (2)$$

La segunda integral es conocida:

$$\int \frac{1}{\sqrt{x^2-1}} = L\,(x + \sqrt{x^2-1}) \qquad (3)$$

Sustituimos (2) i (3) en (1):

$$\int \sqrt{x^2-1}\; d\,x = x\sqrt{x^2-1} - \int \sqrt{x^2-1}\; d\,x - L\,(x + \sqrt{x^2-1});$$

trasponemos la nueva integral i reducimos:

$$\int \sqrt{x^2-1}\; d\,x = \frac{1}{2}\,x\sqrt{x^2-1} - \frac{1}{2}\,L\,(x + \sqrt{x^2-1})$$

5. $$\int \sqrt{1-x^2}\; d\,x = \int \frac{d\,x}{\sqrt{1-x^2}} - \int \frac{x^2}{\sqrt{1-x^2}}\; d\,x$$

La primera integral es conocida:

$$\frac{d\,x}{\sqrt{1-x^2}} = \text{arc sen } x;$$

i la segunda se integra por partes:

Sea

$$u = x,\; d\,v = \frac{x\,d\,x}{\sqrt{1-x^2}}$$

$$\therefore\; d\,u = d\,x,\;\; v = -\sqrt{1-x^2}$$

$$\therefore \int x \frac{x\,dx}{\sqrt{1-x^2}} = -x\sqrt{1-x^2} + \int\sqrt{1-x^2}\,dx$$

Despues de reemplazar, trasponer i reducir, se llega a

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\text{arc sen}\,x + x\sqrt{1-x^2}\right)$$

6. $$\int \sqrt{a-x^2}\,dx = a^2 \int \sqrt{1-\left(\frac{x}{a}\right)^2}\,d\,\frac{x}{a}$$

$$= \frac{a^2}{2}\left(\text{arc sen}\,\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2-x^2}\right.$$

7. $$\int \sqrt{a^2-x^2}\,dx = a^2 \int \sqrt{1-\left(\frac{x}{a}\right)^2}\,.\,a^2\,d\,\frac{x}{a}$$

$$= \frac{a^2}{2}\,\text{arc sen}\,\frac{x}{a} + \frac{1}{2}\,x\sqrt{a^2-x^2}$$

(Cox, 42)

8. $\int \sqrt{A\,x^2+2\,B\,x+C}\,d\,x$. (Tannery, 523)

Sea

$$u = \sqrt{A\,x^2+2\,B\,x+C} \quad \therefore \quad d\,u = \frac{A\,x+B}{\sqrt{A\,x^2+2\,B\,x+C}}\,d\,x$$

i sea

$$v = x + \frac{B}{A} \qquad \therefore \quad d\,v = d\,x.$$

Reemplazando i designando por T el subradical:

$$\int \sqrt{T}\, dx = \frac{1}{A} (Ax + B) \sqrt{T} - \frac{1}{A} \int \frac{(x + B)^2}{\sqrt{T}} dx$$

Pero

$$(Ax + B)^2 = A(T) + B^2 - AC$$

$$\therefore \frac{(Ax + B)^2}{\sqrt{T}} = \frac{AT}{\sqrt{T}} + \frac{B^2 - AC}{\sqrt{T}} = A\sqrt{T} + \frac{B^2 - AC}{\sqrt{T}}$$

$$\int \sqrt{T}\, dx = \frac{1}{A} (Ax + B) \sqrt{T} - \frac{1}{A} \int A\sqrt{T}$$

$$- \frac{1}{A} \int \frac{B^2 - AC}{\sqrt{T}} dx.$$

Trasponemos la nueva integral i reducimos:

$$\int \sqrt{T}\, dx = \frac{1}{2A} (Ax + B) \sqrt{T} - \frac{1}{2A} \int \frac{B^2 - AC}{\sqrt{T}} dx.$$

La nueva integral es conocida.

9. $\int \frac{Lx}{x} dx$. Sea $\left| \begin{array}{ll} u = Lx & \therefore du = \frac{dx}{x} \\ dv = \frac{dx}{x} & \therefore v = Lx \end{array} \right.$

$$y = L^2 x - \int Lx \frac{dx}{x} = \frac{1}{2} L x.$$

10. $\int \sec^3 x\, d\, x$. Sea $\left| \begin{array}{l} u = \sec x \therefore d\, u = \sec x \operatorname{tg} x\, d\, x \\ d\, v = \sec^2 x\, d\, x \therefore v = \operatorname{tg} x. \end{array} \right.$

$$\int \sec^3 x\, d\, x = \sec x \operatorname{tg} x - \int \sec x \operatorname{tg}^2 x\, d\, x$$

$$= \operatorname{sen} x \operatorname{tg} x - \int \sec^3 x\, d\, x + \int \sec x\, d\, x$$

Trasponemos

$$\int \sec^3 x\, d\, x$$

i reducimos:

$$\int \sec^3 x\, d\, x = \frac{1}{2}\left(\sec x \operatorname{tg} x + L\sqrt{\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}}\right)$$

11. $\int x^m L\, x\, d\, x$. $\left| \begin{array}{l} u = L\, x \therefore d\, u = \dfrac{d\, x}{x} \\ d\, v = x^m\, d\, x \therefore v = \dfrac{x^{m+1}}{m+1} \end{array} \right.$

$$\int x^m L\, x\, d\, x = \frac{x^{m+1}}{m+1} L\, x - \int \frac{x^{m+1}}{(m+1)\, x}\, d\, x$$

$$= \frac{x^{m+1}}{m+1}\left(L\, x - \frac{1}{2}\right)$$

(Davies, Peck, 177)

70. *Eliminacion por reduccion.*—Si despues de aplicar la fórmula

$$\int u\, d\, v = u\, v - \int v\, d\, u,$$

encontramos por medio de trasformaciones o descomposiciones que

$$\int u\,d\,v = u_1\,v_1 + \int v\,d\,u,$$

la adicion de las integrales anteriores nos dará:

$$2\int u\,d\,v = u\,v + u_1\,v_1$$

$$\therefore \int u\,v = \frac{1}{4}(u\,v + u_1\,v_1)$$

De este modo se elimina la nueva integral, por reduccion.

1. $\int \operatorname{sen}^2 x\,d\,x$. Sea $\left|\begin{array}{ll} u = \operatorname{sen} x & \therefore\ d\,u = \cos x\,d\,x \\ d\,v = \operatorname{sen} x\,d\,x & \therefore\ v = -\cos x \end{array}\right.$

$$\therefore \int \operatorname{sen}^2 x\,d\,x = -\operatorname{sen} x \cos x + \int \cos^2 x\,d\,x;$$

ademas,

$$\int \operatorname{sen}^2 x\,d\,x = \int d\,x - \int \cos^2 x\,d\,x$$

Sumando, desaparece la nueva integral:

$$2\int \operatorname{sen}^2 x\,d\,x = x - \operatorname{sen} x \cos x$$

$$\therefore \int \operatorname{sen}^2 x\,d\,x = \frac{1}{2}(x - \operatorname{sen} x \cos x)$$

2. $\int \sqrt{a^2 - x^2}\,d\,x$. Integramos por partes:

Sea

$$\left| \begin{array}{ll} u = \sqrt{a^2 - x^2} & \therefore\ d\,u = \dfrac{-x\,d\,x}{\sqrt{a^2 - x^2}} \\ d\,v = d\,x & \therefore\ v = x \end{array} \right.$$

$$\int \sqrt{a^2 - x^2}\,d\,x = x\,\sqrt{a^2 - x^2} + \int \frac{x^2\,d\,x}{\sqrt{a^2 - x^2}}\,;$$

pero,

$$\int \sqrt{a^2 - x^2}\,d\,x = a^2 \int \frac{d\,x}{\sqrt{a^2 - x^2}} - \int \frac{x^2\,d\,x}{\sqrt{a^2 - x^2}}.$$

Sumamos i dividimos por 2:

$$\sqrt{a^2 - x^2}\,d\,x = x\,\sqrt{a^2 - x^2} + a^2 \text{ arc sen } \frac{x}{a}$$

(Claudel, 1129)

3. $$\int \sqrt{x^2 + a^2}\,d\,x = x\,\sqrt{x^2 + a^2} - \int \frac{x^2 d\,x}{\sqrt{x^2 + a^2}}\,,$$

$$\int \sqrt{x^2 + a^2}\,d\,x = a^2 \int \frac{d\,x}{\sqrt{x^2 + a^2}} - \int \frac{x^2 d\,x}{\sqrt{x^2 + a^2}}\,:$$

Sumamos:

$$\int \sqrt{x^2 + a^2}\,d\,x = \frac{x\,\sqrt{a^2 + a^2}}{2} + \frac{a^2}{2} L\,[x + \sqrt{x^2 + a^2}]$$

4. $$\int \sqrt{x^2 + a^2}\, d\, x = -\int \frac{x^2}{\sqrt{x^2 - a^2}}\, d\, x + x\, \sqrt{x^2 - a^2},$$

$$\int \sqrt{x^2 - a^2}\, d\, x = \int \frac{x^2}{\sqrt{x^2 - a^2}}\, d\, x - a^2 \int \frac{d\, x}{\sqrt{x^2 - a^2}}:$$

Sumamos:

$$\int \sqrt{x^2 - a^2}\, d\, x = \frac{1}{2}\, x\, \sqrt{x^2 - a^2} - \frac{1}{2}\, a^2\, L[x + \sqrt{x^2 - a^2}]$$

(Todhunter, 11)

5. $$\int \frac{x\, e^{\text{arc sen } x}}{\sqrt{1 - x^2}}\, d\, x.$$ (Brahy, 33)

Sea

$$u = e^{\text{arc sen } x} \qquad \therefore\ d\, u = \frac{e^{\text{arc sen } x}\, d\, x}{\sqrt{1 - x^2}}$$

$$d\, v = \frac{x}{\sqrt{1 - x^2}} \qquad \therefore\ v = -\sqrt{1 - x^2}$$

$$\therefore\ y = -\sqrt{1 - x^2}\, e^{\text{arc sen } x} + e^{\text{arc sen } x}\, d\, x \qquad (1)$$

Sea ahora:

$$u = x \qquad \therefore\ d\, u = d\, x$$

$$d\, v = \frac{e^{\text{arc sen } x}}{\sqrt{1 - x^2}}\, d\, x \qquad v = e^{\text{arc sen } x}$$

$$\therefore\ y = x\, e^{\text{arc sen } x} - \int e^{\text{arc sen } x}\, d\, x \qquad (2)$$

Sumando i restando las igualdades anteriores (1) i (2), obtenemos:

$$\int \frac{x\, e^{\text{arc sen } x}}{\sqrt{1-x^2}}\, d\, x = \frac{1}{2}\, e^{\text{arc sen } x}\left(x - \sqrt{1-x^2}\right),$$

$$\int e^{\text{arc sen } x}\, d\, x = \frac{1}{2}\, e^{\text{arc sen } x}\left(x + \sqrt{1-x^2}\right).$$

71. *Integracion sucesiva.*—O Doble integracion por partes. —Cuando la nueva integral de

$$\int u\, d\, v = u\, v - \int v\, d\, u,$$

no se puede integrar por ninguno de los dos métodos anteriores, hacemos una segunda integracion por partes de $\int v\, d\, u$.

1. $\int x^2\, e^x\, d\, x$ Sea $\left|\begin{array}{ll} u = x^2 & \therefore\ d\, u = 2\, x\, d\, x \\ d\, v = e^x\, d\, x & \quad v = e^x \end{array}\right.$

$$\int x^2\, e^x\, d\, x = x^2\, e^x - 2 \int x . e^x\, d\, x$$

Integremos de nuevo por partes:

Sea $\left|\begin{array}{ll} u = x & \therefore\ d\, u = d\, x \\ d\, v = e^x\, d\, x & \quad v = e^x \end{array}\right.$

$$\int x^2\, e^x\, d\, x = x^2\, e^x - 2\left(x\, e^x - \int e^x\, d\, x\right)$$

$$= x^2 e^x - 2 x e^x + 2 e^x$$

$$= e^x (x^2 - 2 x + 2)$$

2. $\int x^3 e^x d x.$ $\quad \left| \begin{array}{ll} u = x^3 & \therefore \; d u - 3 x^2 d x \\ d v = e^x d x & \therefore \; v = e^x \end{array} \right.$

$$y = x^3 e^x - 3 \int x^2 e^x d x.$$

Se reduce al ejemplo anterior:

$$y = x^3 e^x - 3 x^2 e^x + 6 x e^x - 6 x^6$$

(De Comberousse, 691)

3. $\int x^2 a^x d x.$ $\quad \left| \begin{array}{ll} u = x^2 & \therefore \; d u = 2 x d x \\ d v - a^x d x & \therefore \; v = \dfrac{a^x}{L a} \end{array} \right.$

$$y = \frac{x^2 a^x}{L a} - \frac{2}{L a} \int x^{a x} d x$$

$$\left| \begin{array}{ll} u = x & \therefore \; d u = d x \\ d v = a^x d x & v = \dfrac{a^x}{L a} \end{array} \right.$$

$$\int x^2 a^x d x = \frac{x^2 a^x}{L a} - \frac{2}{L a} \left(\frac{x a^x}{L a} - \frac{a^x}{L^2 a} \right)$$

$$= \frac{a^x}{L a} \left(x - \frac{1}{L a} \right.^{2}$$

4. $x L^2 x \, d x.$ | $u = L^2 x \quad \therefore \quad d u = 2 L x \frac{d x}{x}$

$d v = x \, d x \quad \therefore \quad v = \frac{1}{2} x^2$

$$y = \frac{1}{2} x^2 L^2 x - \int x L x \, d x.$$

$$= \frac{1}{2} x^2 L^2 x - \frac{1}{2} x^2 L x + \frac{1}{4} x^2$$

5 $\int e^x \cos x \, d x.$ | $u = \cos x \quad \therefore \quad d u = -\operatorname{sen} x \, d x$

$d v = e^x d x \quad \therefore \quad v = e^x$

$$\int e^x \cos x \, d x = e^x \cos x + \int e^x \operatorname{sen} x \, d x. \qquad (1)$$

| $u = \operatorname{sen} x \quad \therefore \quad d u = \cos x \, d x$

$d v = e^x d x \quad \therefore \quad v = e^x$

$$\int e^x \operatorname{sen} x \, d y = e^x \operatorname{sen} x - \int e^x \cos x \, d x \qquad (2)$$

De las igualdades (1) i (2), sacamos:

$$\int e^x \cos x \, d x = \frac{1}{2} e^x (\operatorname{sen} x + \cos x)$$

$$\int e^x \operatorname{sen} x \, d x = \frac{1}{2} e^x (\operatorname{sen} x - \cos x)$$

6. $\int e^{a x} \cos b x \, d x$ | $u = \cos b x \quad \therefore \quad d u = -\operatorname{sen} b x . b \, d x$

$d v = e^{a x} d x \quad \therefore \quad v = \frac{e^{a x}}{a}$

$$\int e^{ax}\cos bx\, dx = \frac{1}{a}\cos bx \,.\, e^{ax} + \frac{b}{a}\int e^{ax}\operatorname{sen} bx\, dx.$$

$$\left|\begin{array}{ll} u = \operatorname{sen} bx & \therefore\; du = b\cos bx\, dx \\ dv = e^{ax}\, dx & \therefore\; v = \frac{1}{a}\, e^{ax} \end{array}\right.$$

$$\int e^{ax}\operatorname{sen} bx\, dx = \frac{1}{a}\operatorname{sen} bx \,.\, e^{ax} + \frac{b}{a}\int e^{ax}\cos bx\, dx.$$

Como en el ejercicio anterior, sacamos:

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\operatorname{sen} bx)}{a^2+b^2}$$

$$\int e^{ax}\operatorname{sen} bx\, dx = \frac{e^{ax}(a\operatorname{sen} bx - b\cos bx)}{a^2+b^2}$$

(Sturm, 355)

7. $$\int x \,.\, \operatorname{arc\,sen} \frac{1}{2}\sqrt{\frac{2a-x}{a}} \,.\, dx.$$

Sea

$$u = \operatorname{arc\,sen} \frac{1}{2}\sqrt{\frac{2a-x}{a}} = \operatorname{arc\,sen} z.$$

$$z = \frac{1}{2}\sqrt{\frac{1a-x}{a}},\quad z^2 = \frac{2a-x}{4a},\quad \sqrt{1-z^2} = \frac{1}{2a}\sqrt{2a+x}.$$

$$du = d\left(\operatorname{arc\,sen}\frac{1}{2}\sqrt{\frac{2a-x}{a}}\right) = \frac{dz}{\sqrt{1-z^2}} = \frac{1}{2\sqrt{4a^2-x^2}}\, dx.$$

$$d\,v = x\,d\,x \qquad \therefore \qquad v = \frac{1}{2}\,x^2$$

$$\therefore \int u\,d\,v = \frac{1}{2}\,x^2 \operatorname{arc\,sen} \frac{1}{2}\sqrt{\frac{2ax-x}{a}} + \frac{1}{4}\int \frac{x^2}{\sqrt{4\,a^2 - x^2}}\,d\,x$$

Integrando por partes la nueva integral, se obtiene:

$$y = \frac{1}{2}\,x^2 \operatorname{arc\,sen} \sqrt{\frac{2\,a\,x - x}{a}} + \frac{1}{2}\,a^2 \operatorname{arc\,sen} \frac{x}{2\,a}$$

$$- \frac{1}{8}\,x\,\sqrt{4\,a^2 - x^2}$$

8. $$\int \frac{x^2 \operatorname{arc\,tg} x}{1+x^2}\,d\,x = \int x^2 \operatorname{arc\,tg} x\,d\,(\operatorname{arc\,tg} x)$$

Sea

$$u = x^2 \qquad \therefore \qquad d\,u = 2\,x\,d\,x$$

$$d\,v = \operatorname{arc\,tg} x\,d\,(\operatorname{arc\,tg} x) \qquad \therefore \qquad v = \frac{1}{2}\operatorname{arc}^2 \operatorname{tg} x$$

$$\int u\,d\,v = \frac{1}{2}\,x^2 \operatorname{arc}^2 \operatorname{tg} x - \int x \operatorname{arc}^2 \operatorname{tg} x\,d\,x.$$

Sea

$$u = x . \operatorname{arc\,tg} x \qquad \therefore \qquad d\,u = \frac{x\,d\,x}{1+x^2} + \operatorname{arc\,tg} x\,d\,x$$

$$d\,v = \operatorname{arc\,tg} x\,d\,x \qquad \therefore \qquad v = x \operatorname{arc\,tg} x - L\,(1+x^2)^{\frac{1}{2}}$$

Sustituyendo i reduciendo, resulta:

$$y = \left(x - \frac{1}{2}\operatorname{arc\,tg} x\right)\operatorname{arc\,tg} x - L\sqrt{1+x^2}$$

(Grégory, 257)

9. $$\int \frac{e^{\operatorname{arc\,tg} x}}{(1+x^2)^{\frac{3}{2}}}\,dx.$$

Sea

$$u = e^{\operatorname{arc\,tg} x} \quad \therefore \quad du = \frac{e^{\operatorname{arc\,tg} x}}{1+x^2}\,dx$$

$$dv = \frac{dx}{\sqrt{(1+x^2)^3}} \quad \therefore \quad v = \int \frac{x^{-3}\,dx}{(x^{-3}+1)^{\frac{3}{2}}}$$

$$= \frac{x}{\sqrt{1+x^2}}$$

$$y = \frac{x\,e^{\operatorname{arc\,tg} x}}{(1+x^2)^{\frac{3}{2}}} - \int \frac{x \,.\, e^{\operatorname{arc\,tg} x}}{(1+x^2)^{\frac{3}{2}}}\,dx$$

Sea

$$u = e^{\operatorname{arc\,tg} x} \quad \therefore \quad du = \frac{e^{\operatorname{arc\,tg} x}}{1+x^2}\,dx$$

$$dv = \frac{x\,dx}{(1+x^2)^{\frac{3}{2}}} \quad \therefore \quad v = -\frac{1}{\sqrt{1+x^2}}$$

$$y = \frac{x+1}{2\sqrt{1+x^2}} e^{\text{arc tg } x}$$

(Brahy, 35)

72. *Reducciones sucesivas.*—Es el caso jeneral de la doble integracion por partes. Se emplea este método en la integracion de la potencia n^a de una funcion, tal como:

$$d\, y = (f\, x)^n\, d\, x:$$

hacemos

$$u = (f\, x)^n \quad \therefore \quad d\, u = n\, (f\, x)^{n-1}\, f'\, (x)\, d\, x$$

$$d\, v = d\, x \quad \therefore \quad v = x;$$

sustituimos en la fórmula

$$\int u\, d\, v = u\, v - \int v\, d\, u$$

i resulta:

$$\int (f\, x)^n\, d\, x = n \int (f\, (x))^{n-1}\, f\, (x)\, d\, x.$$

De este modo se ha conseguido rebajar en una unidad la potencia. Haciendo n − 1 integraciones por partes sucesivas, reducimos al fin la potencia a una forma elemental i conocida.

El mismo método se aplica a potencias de la forma

$$(f\, x)^n\, (\phi\, x)^m\ ,$$

1. $\int L^n x \, d x$. Sea $\left| \begin{array}{ll} u = L^n x & \therefore \; d u = n \, L^{n-1} x \dfrac{d x}{x} \\ d v = d x & \therefore \; v = x \end{array} \right.$

$$\int L^n x \, d x = x \, L^n x - n \int L^{n-1} x \, d x$$

Es evidente que la nueva integral es de forma mas elemental que la primitiva.

En lugar de hacer con la nueva integral una segunda integracion por partes, basta suponer sucesivamente $n=1, 2, 3, \ldots$:

$n=1, \quad \int L \, x \, d x = x \, (L \, x - 1)$

$n=2, \quad \int L^2 x \, d x = x \, (L^2 x - 2 \, L \, x + 2)$

$n=3, \quad \int L^3 x \, d x = x \, (L^3 x - 3 \, L^2 x + 6 \, L \, x - 6)$

$ni=4, \quad \int L^4 x \, d x = x \, (L^4 x - 4 \, L^3 x + 12 \, L^2 x - \ldots \ldots)$

en jeneral

$$\int L^n x \, d x = x \, (L^n x - n \, L^{n-1} x + n \, (n-1) \, L^{n-2} x \ldots \ldots)$$

(Brahy, 36)

2. $\int \frac{1}{L x} d x$. Sea $\left| \begin{array}{ll} u = L^{-1} x & \therefore \; d u = - L^{-2} x \dfrac{d x}{x} \\ d v = d x & \therefore \; v = x \end{array} \right.$

$$\int \frac{L x}{x} d x = x \, L^{-1} x + \int \frac{d x}{L^2 x}$$

Sea $\left| \begin{array}{l} u = L^{-2} x \quad \therefore \quad d\,u = -2\,L^{-3} x \dfrac{d x}{x} \\ d\,v = d\,x \quad \therefore \quad v = x \end{array} \right.$

$$\int \frac{d\,x}{L^2 x} = x\,L^{-2} x + 2!\int L^{-3} x\; d\,x.$$

Procediendo de igual manera, se obtiene la serie:

$$\int \frac{1}{Lx}\, d\,x = x\,(L^{-1} x + L^{-2} x + 2!\; L^{-3} x + \ldots$$

$$+ n!\,L^{-(n+1)} x + \ldots,$$

(Peirce, 57)

3 $\int x^2 \cos x\; d\,x = \int x^2\, d \operatorname{sen} x = x^2 \operatorname{sen} x - 2\int x \operatorname{sen} x\; d\,x,$

$$\int x \operatorname{sen} x\; d\,x = -\int x\; d \cos x = -x \cos x + \int \cos x\; d\,x$$

$$= -\,x \cos x + \operatorname{sen} x.$$

Sustituyendo este valor en la primera igualdad, se tendrá:

$$\int x^2 \cos x\; d\,x = x^2 \operatorname{sen} x + 2\,x \cos x - 2 \operatorname{sen} x.$$

(Sturm, 315)

4. $\int x^3 e^x\, d\,x = x^3 e^x - \int e^x \,.\, 3\,x^2\, d\,x$

$$= x^3 e^x - 3\,x^2\, e^x - \int e^x \,.\, 6\,x\; d\,x$$

$$= x^3 e^x - 3\,x^2\, e^x + 6\,x\, e^x - 6\, e^x$$

$$= e^x (x^3 - 3x^2 + 6x - 6)$$

(De Comberousse, 692)

5. $\int \frac{dx}{(1+x^2)^n}$. Siendo $d\,L\,v = \frac{dv}{v}$,

la fórmula

$$\int u\,dv = uv - \int v\,du$$

se puede escribir, como lo hizo Cauchy:

$$\int uv\,d\,L\,v = uv - \int uv\,d\,L\,u,$$

Suponiendo que *u* i *v* son proporcionales a las potencias x^2 i

$$\frac{1+x^2}{x^2};$$

tendremos:

$$d\,L\,\frac{1+x^2}{x^2} = 2\left(\frac{x}{1+x^2} - \frac{1}{x}\right)dx = -\frac{2\,dx}{x(1+x^2)}$$

$$\therefore \int \frac{dx}{(1+x^2)^n} = \int \frac{-x\,(1+x^2)^{-n+1}}{2}\,d\,L\,\frac{1+x^2}{x^2}$$

$$= \frac{x}{2\,(n-1)\,(1+x^2)^{n+1}} + \frac{2n-3}{2n-2}\int \frac{dx}{(1+x^2)^{n-1}}$$

(Moigno, II, 32)

73. *Fórmulas de reduccion.*—Los métodos de los números anteriores nos conducen naturalmente a fórmulas que, como la última, permiten integrar cualquiera potencia de una funcion. Tales son las fórmulas de reduccion, de las cuales daremos a conocer las mas importantes.

1. *Diferenciales binomias.*—Son espresiones de la forma

$$x^m (a+b\,x^n)^p\,d\,x$$

en la que *m*, *n*, *p* son números conmensurables, i *a*, *b* costantes diferentes de cero.

Para integrar dicha diferencial hai que disminuir sus esponentes, i las fórmulas resultantes llámanse *de reduccion.*

1.º *Rebajar el esponente p.*

Cambiemos m en m−1 i sea $a+b^n=X$; tendremos que integrar

$$y=\int x^{m-1} X^p\,d\,x.$$

Sea

$$\left|\begin{array}{lll} u=X^p & \therefore & d\,u=p\,X^{p-1}\,d\,X, \\ d\,v=x^{m-1}\,d\,x & \therefore & v=\dfrac{x^m}{m}: \end{array}\right.$$

$$y=\frac{x^m X^p}{m}-\int\frac{x^m}{m}\,p\,X^{p-1}\,d\,X.$$

Aquí

$$d\,X=d\,(a+b\,x^n)=b\,n\,x^{n-1}\,d\,x,$$

$$y=\frac{x^m X^p}{m}-\frac{b\,n\,p}{m}\int x^{m+n-1}X^{p-1}\,d\,x.$$

Tal es la fórmula pedida.

2.º *Rebajar el esponente m.*—Traspongamos el término negativo de la fórmula anterior:

$$\int x^{m+n-1} X^{p-1}\, d\,x = \frac{x^m X^p}{b\,n\,p} - \frac{m}{b\,n\,p}\int x^{m-1} X^p\, d\,x$$

Cambiemos m en m—n, p en p+1:

$$\int x^{m-1} X^p\, d\,x = \frac{x^{m-n} X^{p+1}}{b\,n\,(p+1)} - \frac{m-n}{b\,n\,(p+1)}\int x^{m-n-1} X^{p+1} d\,x$$

3.º *Rebajar p* i *m.*—Puesto que

$$X^p = (a+b\,x^n)^p = X^{p-1}(a+b\,x^n),$$

tendremos que

$$\int x^{m-1} X^p\, d\,x = a\int x^{m-1} X^{p-1}\, d\,x + b\int x^{m+n-1} X^{p-1}\, d\,x$$

i en el caso 2.º anterior encontramos que:

$$\int x^{m+n-1} X^{p-1}\, d\,x = \frac{x^m X^p}{b\,n\,p} - \frac{n}{b\,n\,p}\int x^{m-1} X^p\, d\,x.$$

Reemplazamos este valor:

$$\int x^{m-1} X^p\, d\,x = \frac{x^m X^p}{b\,n\,p} - \int\frac{n}{b\,n\,p}\int x^{m-1} X^p\, d\,x$$

$$+ a\int x^{m-1} X^{p-1}\, d\,x.$$

Como aquí aparece dos veces la integral primitiva, reducimos i despejamos:

$$\int x^{m-1} X^p \, d\, x = \frac{x^m X^p}{m+n\,p} + \frac{a\,n\,p}{m+n\,p} \int x^{m-1} X^{p-1} \, d\, x.$$

Estas diferenciales son integrables cuando sean enteros

$$p, q \quad y \quad \frac{m+1}{n} - 1.$$

Daremos algunos ejemplos.

i. $$\int \frac{x^2\, d\, x}{(a+b\,x^2)^{\frac{3}{2}}} = -\frac{x}{b\sqrt{a+b\,x^2}} + \frac{1}{b}\int \frac{d\,x}{\sqrt{a+b\,x^2}}$$

ii. $$\int \frac{x^m\, d\, x}{\sqrt{c^2 - x^2}} = -\frac{x^{m-1}\sqrt{c^2 - x^2}}{m}$$

$$+ \frac{m-1}{m} c^2 \int \frac{x^{m-2}\, d\, x}{\sqrt{c^2 - x^2}}.$$

iii. $$\int \frac{d\,x}{x^m \sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{(m-1)\, a^2\, x^{m-1}}$$

$$-\frac{m-2}{(m-1)a^2} \int \frac{d\,x}{x^{m-1}\sqrt{a^2 + x^2}}$$

(Todhunter, Int. Calc., 42/45)

En la integracion de las funciones se darán a conocer las diferentes formas que tienen las diferenciales binomias.

2. $\int x^n . e^{ax} dx$. Sea $\left| \begin{array}{ll} u = x^n & \therefore du = n x^{n-1} dx \\ dv = e^{ax} dx & \therefore v = \frac{1}{a} e^{ax} \end{array} \right.$

$$y = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

I. Sea $a = 1$,

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$$

II. Sea $n = 1$

$$\int x e^x dx = x e^x - e^x$$

III. $n = 2$: $\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$

$$= x^2 e^x - 2 (x e^x - e^x)$$

En jeneral:

IV. $\int x^n e^x dx = e^x [x^n - n x^{n-1} + n(n-1) x^{n-2} - \dots]$

V. Sea $n = 4$:

$$\int e^{ax} x^4 dx = e^{ax} \left(\frac{x_4}{a} - \frac{4x^3}{a^2} + \frac{12x^2}{a^3} - \frac{24x}{a^4} + \frac{4!}{a^5} \right)$$

VI. Sea $n = 5$, $a = -1$

$$\int \frac{x^5}{e} dx = -e^{-x} (x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120)$$

VI. Sea n negativo:

$$\int \frac{e^{ax}}{x^n} dx = -\frac{e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}}{x^{n-1}} dx$$

VIII. Sea x= —1;

$$\int \frac{e^{ax}}{x} dx = L\, x + a\, x + \frac{a^2 x^2}{1.2^2} + \frac{a^3 x^3}{1.2.3^2} + \ldots$$

IX. Sea n=3, a=1

$$\int \frac{e^x}{x^3} dx = -\frac{e^x}{2x^2}(1+x) + \frac{1}{2}\int \frac{e^x dx}{x}$$

(Grégory, 276-277)

3. $\int x^m L^n x\, dx.$ (Grégory, 277)

Sea

$$u = L^n x \quad \therefore \quad du = n\, L^{n-1} x \frac{dx}{x}$$

$$dv = x^m dx \quad \therefore \quad v = \frac{x^{m+1}}{m+1}$$

$$y = \frac{x^{m+1} L^n x}{m+1} - \frac{n}{m+1}\int x^m L^{n-1} x\, dx$$

I. Sea n=1,

$$\int x^m L\, x\, dx = \frac{x^{m+1} L\, x}{m+1} - \frac{1}{m+1} \cdot \frac{x^{m+1}}{m+1}$$

II. Sea $n=2$,

$$\int x^m L^2 x\, d\, x = \frac{x^{m+1} L^2 x}{m+1} - \frac{2}{m+1} \cdot$$

$$- \frac{x^{m+1} L\, x}{m+1} + \frac{x^{m+1}}{(m+1)^2}$$

III. Sea $m=3$, $u=2$

$$\int x^3 L^2 x\, d\, x = \frac{1}{4} x^4 \left(\log^2 x - \frac{1}{2} L\, x + \frac{1}{8}\right)$$

IV. Sea $m=1$, $n=3$

$$\int x\, L^3 x\, d\, x = \frac{1}{2} x^2 \left(L^3 x - \frac{3}{2} L^2 x - \frac{3}{4}\right)$$

4. $\int \operatorname{sen}^m x \cos^n x\, d\, x$. Esta es una de las integrales que mas aplicaciones tiene.

I. Hacemos aljebraica la derivada, poniendo

$$z = \operatorname{sen} x \quad \therefore \quad \sqrt{1-z^2} = \cos x,\ d\, x = \sqrt{1-z^2}\, d\, z$$

$$\operatorname{sen}^m x \cos^n x\, d\, x = z^m (1-z^2)^{\frac{n-1}{2}} d\, x$$

II. Integracion por partes.

Sea

$$u = \cos^{n-1} x \qquad d\, x = -(n-1) \cos^{n-2} x \cdot \operatorname{sen} x\, d\, x$$

$$d\,v = \operatorname{sen}^m x \cos x\, d\,x \quad \therefore \quad v = \frac{\operatorname{sen}^{m+1} x}{m+1}$$

$$\therefore \int \operatorname{sen}^m x \cos^n x\, d\,x$$

$$= \frac{\operatorname{sen}^{m+1} x \,.\, \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \operatorname{sen}^{m+2} x \cos^{n-1} x\, d\,x.$$

Para disminuir el nuevo esponente de sen, empleamos la trasformacion que sigue:

$$\operatorname{sen}^{m+2} x = \operatorname{sen}^m x\ (1 - \cos^2 x):$$

$$\therefore \int \operatorname{sen}^m x \cos^n x\, d\,x$$

$$= \frac{\operatorname{sen}^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \operatorname{sen}^m x \cos^{n-1} x\, d\,x$$

$$- \frac{n-1}{m+1} \int \operatorname{sen}^m x \cos^n x\, d\,x.$$

Aquí aparece la integral primitiva; trasponemos i despejamos:

$$\int \operatorname{sen}^m x \cos^n x\, d\,x = \frac{\operatorname{sen}^{m+1} x \cos^{n-1} x}{m+n}$$

$$= \frac{n-1}{m+n} \int \operatorname{sen}^m x \cos^{n-2} x\, d\,x$$

III. Haciendo $u = \operatorname{sen}^{m-1} x$, $d\,v = \cos^n x \operatorname{sen} x\, d\,x$, se obtiene m rebajado en 2 unidades:

$$\int \operatorname{sen}^m x \cos^n x \, dx = -\frac{\operatorname{sen}^{m-1} x \cos^{n+1} x}{m+n}$$

$$= \frac{m-1}{m+n} \int \operatorname{sen}^{m-2} x \cos^n x \, dx$$

IV. Sea $n=0$, la segunda fórmula nos da:

$$\int \operatorname{sen}^m x \, dx = -\frac{\operatorname{sen}^{m-1} x \cos x}{m} + \frac{m-1}{m} \int \operatorname{sen}^{m-2} x \, dx.$$

V. Sea $m=0$, la primera nos dá:

$$\int \cos^n x \, dx = \frac{\operatorname{sen} x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

VI. Si m es par:

$$\int \operatorname{sen}^m x \, dx = -\frac{\cos x}{m} \Big[\operatorname{sen}^{m-1} x + \frac{m-1}{m-2} \operatorname{sen}^{m-3} x$$

$$+ \frac{(m-1)(m-3)}{(m-2)(m-4)} \operatorname{sen}^{m-5} x + \ldots + \frac{(m-1)\ldots\ldots 3.1}{(m-2)\ldots\ldots 4.2} \operatorname{sen} x \Big]$$

$$+ \frac{(m-1)\ldots.3.1}{(m-2)\ldots 4.2} \cdot \frac{x}{m} \; .$$

VII. m es negativo, $n=0$:

$$\int \frac{dx}{\operatorname{sen}^m x} = -\frac{\cos x}{(m-1)\operatorname{sen}^{m-1} x} + \frac{m-2}{m-1} \int \frac{dx}{\operatorname{sen}^{m-2} x}$$

VIII. n es negativo:

$$\int \frac{\operatorname{sen}^m x\, d x}{\cos^n x} = -\frac{\operatorname{sen}^{m-1} x}{(m-n)\cos^{n-1} x} + \frac{m-1}{m-n}\int \frac{\operatorname{sen}^{m-2} x\, d x}{\cos^n x}.$$

IX. Cambiando n por —m,

$$\int \operatorname{tg}^m x\, d x = \frac{1}{m-1}\operatorname{tg}^{m-1} x - \int \operatorname{tg}^{m-2} x\, d x$$

X. m=2, n=0:

$$\int \operatorname{sen}^2 x\, d x = -\frac{1}{2}(\operatorname{sen} x \cos x - x)$$

XI. $\int \frac{d x}{\operatorname{sen}^3 x} = -\frac{\cos x}{2 \operatorname{sen}^2 x} + \frac{1}{2} L \operatorname{tg} \frac{x}{2}$

XII. $\int \frac{\operatorname{sen}^4 x}{\cos x} d x = -\frac{\operatorname{sen} x}{3}(\operatorname{sen}^2 x + 3) + L \operatorname{tg}\left(\frac{1}{4}\pi + \frac{1}{2} x\right)$

XIII. $\int \operatorname{sen}^3 x \cos^2 x\, d x = -\frac{1}{5}\cos^3 x\left(\operatorname{sen}^2 x + \frac{2}{3}\right)$

XIV. $\int \frac{d x}{\operatorname{sen} x \cos^4 x} = \frac{1}{3\cos^3 x} + \frac{1}{\cos x} + L \operatorname{tg} \frac{1}{2} x$

XV. $\int \operatorname{tg}^4 x\, d x = \frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x.$

5. $\int x^n \cos x\, d x,$ $\left|\begin{array}{l} u = x^n \\ d v = \cos x\, d x \end{array}\right.$ $\therefore$ $d u = u\, x^{n-1} d x$, $v = \operatorname{sen} x$

$$y = x^n \operatorname{sen} x - n \int x^{n-1} \operatorname{sen} x \, d\,x$$

$$\left| \begin{array}{ll} u = x^{n-1} \quad \therefore & du = (n-1) x^{n-2}\, d\,x \\ d\,v = \operatorname{sen} x \, d\,x \quad \therefore & v = -\cos x \end{array} \right.$$

$$y = x^n \operatorname{sen} x + n\, x^{n-1} \cos x - n\,(n-1) \int x^{n-2} \cos x \, d\,x$$

I. $\int x^2 \cos x \, d\,x = x^2 \operatorname{sen} x + 2\,x \cos x - 2 \operatorname{sen} x$

II. $\int x^3 \cos x \, d\,x = x^3 \operatorname{sen} x + 3\,x^2 \cos x - 6\,x \operatorname{sen} x - 6 \cos x$

III. $\int x \operatorname{sen} x \, d\,x = -x \cos x + \operatorname{sen} x$

IV. $\int x^4 \operatorname{sen} x \, d\,x = -x^4 \cos x + 4\,x^3 \operatorname{sen} x + 12\,x^2 \cos x$

$$-242 \operatorname{sen} x - 24 \cos x$$

74. Integrales principales.—El estudio de los métodos i ejercicios que preceden, habrá dado a conocer al lector la forma i naturaleza de la Integracion, operacion que consiste en trasformar la diferencial propuesta hasta reducirla a una de las fórmulas simples llamadas *fundamentales*.

En el número 75 damos las fórmulas que a nuestro juicio son verdaderamente fundamentales: en realidad no tienen ningun valor práctico sino abstracto i jeneral.

Las fórmulas usuales a que hai que reducir las diferenciales que se quiere integrar, con mas propiedad las llamamos *principales*, en atencion a la importancia que tienen, es decir, a la frecuencia con que se necesitan en la Integracion.

Antes de formar este grupo de fórmulas principales, hemos comparado i discutido cada una de las integrales fundamentales que recomiendan los autores cuyos nombres hemos citado en los ejercicios: algunos, como Roberts, dan hasta 45 de estas fórmulas; otros, como Jordan, reducen su número a 7. Creemos que las 12 fórmulas que siguen son suficientes i abarcan todos los casos que pueden presentarse en la Integracion:

INTEGRALES PRINCIPALES

I. *signo*: $\int \pm dx = \pm x$

II. *coeficiente*: $\int a\,dx = \pm a x$

III. *esponente*: $\int x^n dx = \frac{x^{n+1}}{n+1}$

IV. *Logaritmo*: $\int \frac{1}{x} dx = L x$

V. *arcotanjente*: $\int \frac{1}{1+x^2} dx = \text{arc tg } x$

VI. *Log. radical*: $\int \frac{1}{1-x^2} dx = L \sqrt{\frac{1+x}{1-x}}$

VII. *Log. binomio*: $\int \frac{1}{\sqrt{1+x^2}} dx = L\left(x + \sqrt{1+x^2}\right)$

VIII. *arco seno*: $\int \frac{1}{\sqrt{1-x^2}} dx = \text{arc sen } x$

IX. *esponencial*: $\int e^x dx = e^x$

X. *coseno*: $\int \text{sen } x\, dx = -\cos x$

XI. *seno*: $\int \cos x \, d x = \text{sen}\, x$

XII. *tanjente*: $\int \sec^2 x \, d x = \text{tg}\, x$

Para integrar las demas diferenciales, sean simples o compuestas, se reducen, como ya se dijo, a una de estas doce fórmulas, trasformándolas mediante uno de los métodos ya estudiados i que pueden quedar comprendidos en los cuatro siguientes:

XIII. *Trasformaciones aljebraicas*: $\int f'(x) \, d x = \int d f x$

XIV. *Descomposiciones*: $\int (d u \pm d v) = \int d u \pm \int d v$

XV. *Sustitucion*: $\int f(x) \, d x = \int f(\phi z) \, \phi' z \, d z$

XVI. *Integracion por partes*: $\int u \, d v = u v - \int v \, d u$.

A continuacion van las doce fórmulas anteriores seguidas de las diferentes derivadas que dan como fundamentales los autores citados:

I. $\int \pm d x$.

II. $\int a \, d x$.

III. $\int x^n d x$: $x^{\pm n}$, $x^{\frac{1}{n}}$, $a x^n$, $n x^{n-1}$

$\frac{1}{2\sqrt{x}}$, $(x \pm a)^{\pm u}$, $(a x + b)$

$f x)^u f' x$

IV. $\int \frac{1}{x} dx$: $\pm \frac{1}{x}$, $\frac{a}{x}$, $\frac{1}{x-a}$, $\frac{1}{ax+b}$, $\frac{f'(x)}{f(x)}$

V. $\int \frac{1}{1+x^2} dx$: $-\frac{1}{1+x^2}$, $\frac{1}{a^2+x^2}$

VI. $\int \frac{1}{1-x^2} dx$: $\pm \frac{1}{1-x^2}$, $\pm \frac{1}{a^2-x^2}$, $\pm \frac{1}{x^2-a^2}$

VII. $\int \frac{1}{\sqrt{1+x^2}} dx$: $\pm \frac{1}{\sqrt{x^2 \pm a^2}}$

VIII. $\int \frac{1}{\sqrt{1-x^2}} dx$: $-\frac{1}{\sqrt{1-x^2}}$, $\pm \frac{1}{\sqrt{a^2-x^2}}$,

$$\frac{1}{\sqrt{2x-x^2}},\ \pm \frac{dx}{x\sqrt{x^2-a^2}},\ \frac{dx}{x\sqrt{a^2 \pm x^2}}$$

IX. $\int e^x dx$: e^{nx}, e^{ax+b}, a^x, a^{nx}, $a^x L a$

X. $\int \operatorname{sen} x\, dx$: $\pm \operatorname{sen} a x$.

XI. $\int \cos x\, dx$: $\cos a x$, $\operatorname{tg} x$, $\cot x$, $\sec x$, $\operatorname{cosec} x$.

XII. $\int \sec^2 x\, dx$: $\pm \sec^2 a x$

$$\operatorname{cosec}^2 x,\ \frac{\operatorname{sen} x}{\cos^2 x},\ \frac{\cos x}{\operatorname{sen}^2 x}$$

$$\frac{1}{\operatorname{sen} x \cos x}.$$

75. *Formas fundamentales.*—Las fórmulas principales anteriores se pueden reducir, a su vez, a otros mas simples, que consideramos como fundamentales; tales son las dos siguientes:

$$\int d\,x = x, \qquad \text{(A)}$$

$$\int \frac{1}{x}\, d\,x = L\,x. \qquad \text{(B)}$$

La primera es la definicion misma del Cálculo Integral, i la segunda es el logaritmo natural de la funcion.

Se establecen estas dos fórmulas invirtiendo las reglas de la diferenciacion.

Las llamamos *formas fundamentales*, porque son las mas simples en cuanto a la forma, es decir, son irreductibles o no se pueden descomponer en otras mas sencillas: i porque con su ayuda i con los métodos de Integracion, se integran todas las diferenciales que hasta hoi se han podido integrar.

Conforme con esto, vamos a hacer la integracion de las diferenciales principales, lo que, aun cuando no es mas que un ejercicio de cálculo, servurá para completar los conoci mientos anteriores.

I. $\int \pm\, d\,x.$

Una sencilla frasformacion nos da:

$$\int \pm\, d\,x = \int d\,(\pm\, x) = \pm\, x$$

O, si se quiere, hacemos

$$\pm\, x = u \quad .\cdot. \quad \pm\, d\,x = d\,u$$

i se tendrá

$$\int \pm dx = \int du = u = \pm x.$$

II. $\int a\,dx$.

Como es el caso anterior:

$$\int a\,dx = \int d\,ax = ax$$

$$= \int du = u = ax$$

III. $\int x^n dx$.

Integramos por partes:

Sea

$$u = x^n \quad \therefore \quad du = n\,x^{n-1} dx$$

i sea

$$dv = dx \quad \therefore \quad v = x$$

$$\therefore \int x^n dx = x^n . x - n \int x^{n-1} . x . dx.$$

La nueva integral se traspone i se obtiene:

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

1. $\int x^{\pm n} dx$. Hacemos $\pm n = m$ i se vuelve al ejemplo anterior:

$$\int x^{\pm n}\, dx = \int x^m\, dx = \frac{x^{m+1}}{m+1} = \frac{x^{\pm n+1}}{\pm n+1}.$$

2. $$\int x^{\frac{1}{n}} dx = \int x^m\, dx = \frac{x^{m+1}}{m+1} = \frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}.$$

3. $$\int a\, x^n dx = a\int x^n dx = \frac{a\, x^{n+1}}{n+1}.$$

4. $$\int n\, x^{n-1}\, dx = n\int x^m dx = n.\frac{x^n}{n} = x^n.$$

5. $$\int \frac{1}{2\sqrt{x}} dx = \frac{1}{2}\int x^{-\frac{1}{2}} = \sqrt{x}.$$

6. $$\int (x \pm a)^{\pm n}\, dx = \int (x \pm a)^{\pm n}\, d\,(x \pm a)$$

$$= \frac{(x \pm a)^{\pm n+1}}{\pm n+1}.$$

7. $$\int (a\,x+b)^n d\,x = \frac{1}{a}\int (a\,x+b)^n\, d\,(a\,x+b)$$

$$= \frac{(a\,x+b)^{n+1}}{a\,(n+1)}.$$

8. $$\int (f\,x)^n\, f^1\, x\, d\,x = \int (f\,x)^n d\,(f\,x) = \frac{(f\,x)^{n+1}}{n+1}.$$

IV. $$\int \frac{1}{x}\, d\,x.$$

Forma fundamental:

$$\int \frac{1}{x}\, d\,x = L\,x.$$

9. $$\int \pm \frac{1}{x}\, d\,x = \pm \int \frac{1}{x}\, d\,x = \pm L\,x$$

10. $$\int \frac{a}{x}\, d\,x = a \int \frac{1}{x}\, d\,x = a\,L\,x = L\,(x^a)$$

11. $$\int \frac{1}{x-a}\, d\,x = \int \frac{d\,(x-a)}{x-a} = L\,(x-a)$$

12. $$\int \frac{1}{a\,x+b}\, d\,x = \frac{1}{a} \int \frac{d\,(a\,x+b)}{a\,x+b} = \frac{1}{a}\, L\,(a\,x+b)$$

13. $$\int \frac{f^1\,x}{f\,x}\, d\,x = \int \frac{d\,(f\,x)}{f\,x} = L\,(f\,x).$$

V. $$\int \frac{1}{1+x^2}\, d\,x.$$

La sustitucion trigonométrica nos dá, haciendo

$$x = \operatorname{tg}\theta \quad \therefore \quad d\,x = \sec^2\theta\, d\,\theta,$$

$$1 + x^2 = \sec^2\theta:$$

$$\int \frac{1}{1+x^2}\, d\,x = \int \frac{\sec^2\theta\, d\,\theta}{\sec^2\theta\, d\,\theta} = \int d\,\theta = \theta = \operatorname{arc\,tg} x.$$

14. $\int -\frac{1}{1+x^2} d\,x$. Sea $x = \cot\theta \therefore d\,x = -\operatorname{cosec}^2\theta\, d\,\theta$:

$$\int -\frac{1}{1+x^2} d\,x = \int d\,\theta = \theta = \text{arc cot } x.$$

15. $$\int \pm \frac{1}{a^2+x^2} d\,x = \frac{1}{a} \int \frac{\pm\, d\,\frac{x}{a}}{1+\left(\frac{x}{a}\right)^2} = \begin{cases} \frac{1}{a} \text{ arc tg } \frac{x}{a} \\ \frac{1}{a} \text{ arc cot } \frac{x}{a} \end{cases}$$

VI. $\int \frac{1}{1-x^2} d\,x$.

El método de las fracciones parciales nos conduce a la trasformacion:

$$\int \frac{1}{1-x^2} d\,x = \int \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) d\,x$$

$$= L\sqrt{\frac{1+x}{1-x}}$$

16. $$\int -\frac{1}{1-x^2} d\,x = -L\sqrt{\frac{1+x}{1-x}} = L\sqrt{\frac{1-x}{1+x}}$$

17. $$\int \pm \frac{d\,x}{a^2-x^2} = \pm \frac{1}{a} \int \frac{d\,\frac{x}{a}}{1-\left(\frac{x}{a}\right)^2}$$

$$= \pm \frac{1}{a} L \sqrt{\frac{1+\frac{x}{a}}{1-\frac{x}{a}}} = \pm \frac{1}{a} L \sqrt{\frac{a+x}{a-x}}\,.$$

18. $$\int \pm \frac{1}{x^2-a^2} dx = \mp \int \frac{dx}{a^2-x^2} = \mp \frac{1}{a} L \sqrt{\frac{a+x}{a-x}}$$

VII. $$\int \frac{1}{\sqrt{1+x^2}} dx.$$

Sustitucion aljebraica.

Sea

$$1+x^2 = z^2 \quad \therefore \quad \frac{dx}{z} = \frac{dz}{x}$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \int \frac{dx}{z} = \int \frac{d(x+z)}{z+z}$$

$$= L\left(x + \sqrt{1+x^2}\right)$$

19. $$\int \pm \frac{1}{\sqrt{x^2 \pm a^2}} = \pm \int \frac{dx}{z} = \pm L\left(x \pm \sqrt{x^2+a^2}\right)$$

VIII. $$\int \frac{1}{\sqrt{1-x^2}} dx.$$

Sustitucion trigonométrica:

Sea

$$x = \text{sen } t \quad \therefore \quad dx = \cos t \, dt:$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \int \frac{\cos t \, dt}{\cos t} = \int dt = t = \text{arc sen } x.$$

20. $\int \frac{-1}{\sqrt{1-x^2}}\, dx$. Sea $x = \cos t \therefore dx = -\operatorname{sen} t\, dt$

$$\int \frac{-1}{\sqrt{1-x^2}}\, dx = \int dt = t = \operatorname{arc} \cos x.$$

21. $$\int \pm \frac{1}{\sqrt{a^2-x^2}}\, dx = \int \frac{\pm d\,\frac{x}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \begin{cases} \operatorname{arc\,sen} \frac{x}{a} \\ \operatorname{arc\,sen} \frac{x}{a} \end{cases}$$

22. $$\int \frac{1}{\sqrt{2x-x^2}}\, dx = \int \frac{-d(1-x)}{\sqrt{1-(1-x)^2}}$$

$$= \operatorname{arc} \cos (1-x)$$

23. $$\int \frac{\pm 1}{x\sqrt{x^2-a^2}}\, dx = \int \frac{\pm dx}{x^2 \sqrt{1-\left(\frac{a}{x}\right)^2}}$$

$$= \frac{1}{a} \int \frac{\mp d\,\frac{a}{x}}{\sqrt{1-\left(\frac{a}{x}\right)^2}} = \begin{cases} \frac{1}{a} \operatorname{arc} \cos \frac{a}{x} \\ \frac{1}{a} \operatorname{arc\,sen} \frac{a}{x} \end{cases}$$

$$= \begin{cases} \frac{1}{a} \operatorname{arc\,sec} \frac{x}{a} \\ \frac{1}{a} \operatorname{arc\,cosec} \frac{x}{a} \end{cases}$$

24. $$\int \frac{\pm 1}{x\sqrt{a^2 \pm x^2}} = \pm \frac{1}{a} \int \frac{d\,\frac{a}{x}}{\sqrt{\left(\frac{a}{x}\right)^2 \pm 1}}$$

$$= \mp \frac{1}{a} L\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 \pm 1}\right)$$

$$= \mp \frac{1}{a} L\left(\frac{a}{x} + \sqrt{\frac{a^2 \pm x^2}{x^2}}\right)$$

$$= \mp \frac{1}{a} L\left(\frac{a + \sqrt{a^2 \pm x^2}}{x}\right)$$

IX. $\int e^x dx$. Sea $e^x = z \therefore de^x = e^x dx = dz$

$$\int e^x dx = \int dz = z = e^x$$

25. $\int e^{nx} dx$ Sea $nx = z \therefore dx = \frac{dz}{n}$

$$\int e^{nx} dx = \frac{1}{n} \int e^z dz = \frac{1}{n} e^{nx}.$$

26. $\int e^{ax+b} dx$. Sea $ax + b = z \therefore dx = \frac{dz}{a}$:

$$\int e^{ax+b} dx = \frac{1}{a} \int e^z dz = \frac{1}{a} e^{ax+b}.$$

27. $\int a^x dx = \int e^{x L a} dx = \frac{1}{La} e^{x La} = \frac{1}{La} a^x$.

En esta trasformacion suponemos:

$$e^z = a^x \quad \therefore \quad z = x L a$$

$$\therefore\ a^x = e^{x\,L\,a}$$

28. $\int a^{n\,x} d\,x = \frac{1}{n} \int a^z\,d\,z = \frac{1}{n} \cdot \frac{a^{n\,x}}{L\,a}$.

29. $\int a^x\,L\,a\,d\,x. = L\,a\,\frac{a^x}{L\,a} = a^x$

X. $\int \operatorname{sen} x\,d\,x.$

Segun las ecuaciones de Euler, tendremos:

$$\int \operatorname{sen} x\,d\,x = \int \frac{e^{i\,x} - e^{-i\,x}}{2\,i}\,d\,x$$

$$= \frac{1}{2\,i}\left(\int e^{i\,x} - \int e^{-i\,x}\right) d\,x$$

$$= \frac{1}{2\,i} \cdot \left(\frac{e^{i\,x}}{i} + \frac{e^{i\,x}}{i}\right)$$

$$= -\frac{e^{i\,x} + e^{-i\,x}}{2} = -\cos x$$

30. $\int \operatorname{sen} a\,x\,d\,x = \frac{1}{a} \int \operatorname{sen} a\,x\,d\,(a\,x)$

$$= -\frac{1}{a} \cos a\,x.$$

XI. $\int \cos x\,d\,x = \int \operatorname{sen}\,(90^o - x)\,d\,x$

$$=\int -\operatorname{sen}(90^{\circ}-x)\, d\,(90^{\circ}-x)$$

$$=\cos(90^{\circ}-x)=\operatorname{sen} x$$

31. $\int \cos a x\, d x=\frac{1}{a}\int \cos a x\, d\, a x=\frac{1}{a}\operatorname{sen} a x.$

32. $\int \operatorname{tg} x\, d x=\int \frac{\operatorname{sen} x\, d x}{\cos x}=-\int \frac{d \cos x}{\cos x}=-L \cos x$

33 $\int \cot x\, d x=\int \frac{d \operatorname{sen} x}{\operatorname{sen} x}=L \operatorname{sen} x$

34. $\int \sec x\, d x=\int \frac{d x}{\cos x}=\int \frac{\cos x\, d x}{\cos^{2} x}=\int \frac{d \operatorname{sen} x}{1-\operatorname{sen}^{2} x}$

$$=L \sqrt{\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}}=L \operatorname{tg} \frac{1}{2}\left(\frac{1}{2} \pi+x\right)$$

35. $\int \operatorname{cosec} x\, d x=\int \frac{d x}{\operatorname{sen} x}=\int \frac{-d \cos x}{1-\cos^{2} x}$

$$=-L \sqrt{\frac{1+\cos x}{1-\cos x}}=L \operatorname{tg} \frac{1}{2} x$$

XII. $\int \sec^{2} x\, d x.$

$$\int \sec^{2} x\, d x=\int \frac{1}{\cos^{2} x} d x=\int \frac{\cos^{2} x+\operatorname{sen}^{2} x}{\cos^{2} x} d x$$

$$=\int \frac{\cos x\, d \operatorname{sen} x-\operatorname{sen} x\, d \cos x}{\cos^{2} x}$$

$$= \int d \frac{\text{sen } x}{\cos x} = \text{tg } x$$

36. $\int \sec^2 a\, x\, d\, x = \frac{1}{a} \int \sec^2 a\, x\, d\, a\, x$

$$= \frac{1}{a} \text{tg } a\, x$$

37. $\int \text{cosec}^2 x\, d\, x = \int \sec^2 (90^o - x)\, d\, x$

$$= -\text{tg } (90^o - x) = -\cot x.$$

38. $\int \frac{\text{sen } x}{\cos^2 x}\, d\, x = -\int \cos^{-1} x\, d \cos x$

$$= \cos^{-1} x = \sec x.$$

39. $\int \frac{\cos x}{\text{sen}^2 x}\, d\, x = \int \text{sen}^{-2} x\, d \text{ sen } x$

$$= -\text{sen}^{-1} x = -\text{cosec } x.$$

40. $\int \frac{1}{\text{sen } x \cos x}\, d x = \int \frac{\frac{d\, x}{\cos^2 x}}{\frac{\text{sen } x}{\cos x}} = \int \frac{d \text{ tg } x}{\text{tg } x}$

$$= \text{L tg } x.$$

Valparaiso, Febrero 23 de 1909.

CÁRLOS WARGNY.

AUTORES CONSULTADOS

Appell

Bertrand
Boucharlat
Bossut
Bowser
Brahy
Briot
Byerly

Carnot
Catalan
Cauchy
Claudel
Comberousse
Cournot
Cox

Dariès
Davies
Duhamel

Edwards
Euler

Fourcy
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Freycinet

Greenhill

Grégory

Haag
Hall
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Hermite
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Humbert
Hutton

Jarrett
Jordan

Karnack

Lacroix
Lardner
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Tannery
Timmermans
Tisserand
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