LOS MÉTODOS DE INTEGRACION

POR

CARLOS WARGNY

(Continuacion)

7. $\int (a + b x^m)^n x^{m-1} dx = \dfrac{(a + b x^m)^{n+1}}{b m (n+1)}$ (Brahy, 13)

8. $\int a x^{n-1} dx (b + c x^n)^m = \dfrac{a (b + c x_n)^{m+1}}{c (m+1)}$

(Francoeur, 357)

9. $\int_x (a^m - b x^{m+1})^n x^m dx$

$$= \frac{-1}{b (m+1)} \int (a^m - b x^{m+1})^n d (a^m - b x^{m=1}$$

$$= \frac{(a^m - b x^{m+1})^{n+1}}{b (m+1)(n+1)}$$

Fraccionarias.

10. $\int \dfrac{dx}{x} = \dfrac{1}{\log e} \int \dfrac{\log e}{x} dx = \dfrac{\log x}{\log e}$ (Pauly, 156)

11. $\int_x \frac{1}{1-x} = -\int \frac{d(1-x)}{1-x} = -L(1-x) = L\frac{1}{1-x}$

(Peacock, 270)

12. $\int \frac{dx}{a+bx} = \frac{1}{b}\int \frac{d(a+x)}{a+bx} = \frac{1}{b} L(a+bx).$

13. $\int_x \frac{5}{2x+3} = \frac{5}{2}\int \frac{2dx}{2x+3} = L\sqrt{(2x+3)^5}$

14. $\frac{Adx}{mx+n} = \frac{A}{m}\int \frac{mdx}{mx+n} = \frac{A}{m} L(mx+n)$

(Sonnet, 205)

15. $\int \frac{dx}{1-x^2}$. Esta integral es mui importante.

Se integra fácilmente si observamos que

$$\frac{1}{1-x^2} = \frac{\frac{1}{(1-x)^2}}{\frac{1+x}{1-x}},$$

i que

$$d\,\frac{1+x}{1-x} = \frac{2}{(1-x)^2}\,dx,$$

en consecuencia, hai que introducir el coeficiente 2:

$$\int \frac{dx}{1-x^2} = \int \frac{d\,\frac{1+x}{1-x}}{\frac{1+x}{1-x}} = \frac{1}{2} L\frac{1+x}{1-x} = L\sqrt{\frac{1+x}{1-x}}$$

(Price, II, 21)

16. $\int \frac{dx}{x^2-1} = -\int \frac{dx}{1-x^2} = -L\sqrt{\frac{1-x}{1+x}} = L\sqrt{\frac{1-x}{1+x}}$

17. $$\int_x \frac{dx}{1+a^2x^2} = \frac{1}{a}\int \frac{d\,ax}{1+(ax)^2} = \frac{1}{a} \text{ arc tg } ax$$

18. $$\int \frac{dx}{1+ax^2} = \frac{1}{\sqrt{a}}\int \frac{d\sqrt{a}\,x}{1+(\sqrt{a}\,x)^2} = \frac{1}{\sqrt{a}} \text{ arc tg } \sqrt{a}\,x$$

19. $$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = \frac{1}{2}\,L\,(1+x^2)$$

(Hall, 220)

20. $$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\int_x \frac{2x}{a^2+x^2)} = L\,(a^2+x^2)^{\frac{1}{2}}$$

(Briot, II, 194)

21. $$\int \frac{dx}{x^2+ax} = \int \frac{\dfrac{dx}{(x+a)^2}}{\dfrac{x}{x+a}} = \frac{1}{a}\,L\,\frac{x}{x+a}$$

(Roberts, 10)

22. $$\int_x \frac{x^2}{1+x^3} = \frac{1}{3}\int \frac{3x}{1+x^3}\,dx = L\,\sqrt{1+x^3}$$

23. $$\int_x \frac{5x^3}{3x^4+7} = \int \frac{5}{12}\cdot\frac{12x^3dx}{3x^4+7} = \frac{5}{12}\,L\,(3x^4+7)$$

24. $$\int_x \frac{x^{n-1}}{a+bx^n} = \frac{1}{bn}\int \frac{d(a+bx^n)}{a+bx^n} = \frac{1}{bn}\,L(a+bx^n)$$

(Williamson, II, 3)

25. $$\int_x \frac{Ax+B}{Ax^2+Bx+C} = \frac{1}{2}\,L\,(Ax^2+Bx+C)$$

(J. Tannery, II, 518)

Irracionales.

26. $$\int x\sqrt{x^2+a^2}\,dx = \frac{1}{2}\int (x^2+a^2)^{\frac{1}{2}}\,2x\,dx =$$

$$\frac{1}{2}\int \frac{2}{3}(x^2+a^2)^{\frac{3}{2}}\, d\, x = \frac{1}{3}(x^2+a^2)^{\frac{3}{2}}.$$

(Stone, Ap. 32)

27. $$\int 5\, x^2 \sqrt{c^2 - x^3} = -\frac{5}{3}\int u^{\frac{1}{2}}\, d\, u = \frac{10}{9}\sqrt{(c^2 - x^2)^3}$$

(Hutton, I, 483)

28. $$\int_x 6\, x \sqrt{4\, x^2 + 3} = \frac{1}{2}(4\, x^2 + 3)^{\frac{3}{2}}$$

29. $$\int \frac{d\, x}{\sqrt{1 - a^2 x^2}} = \frac{1}{a}\int [\, 1 - (a\, x)^2\,]^{-\frac{1}{2}}\, d\, (a\, x) =$$

$$\frac{1}{a}\ \text{arc. sen}\ a\, x$$

30. $$\int \frac{d\, x}{(1 - a\, x)^{\frac{1}{2}}} = \frac{1}{\sqrt{a}}\int \frac{d\, \sqrt{a}\, x}{\sqrt{1 - (\sqrt{a}\, x)^2}} =$$

$$\frac{1}{\sqrt{a}}\ \text{arc. sen}\ \sqrt{a}\, x$$

31. $$\int x\, (a^2 + x^2)^{-\frac{1}{2}}\, d\, x = \frac{1}{2}\int (a^2 + x^2)^{\frac{1}{2}}\, d\, (a^2 + x^2) =$$

$$\sqrt{a^2 + x^2}$$

32. $$\int \frac{x\, d\, x}{\sqrt{a^2 - b\, x^2}} = -\frac{1}{2\, b}\int (a^2 - b\, x^2)^{-\frac{1}{2}}\, d(-2bx\, dx)$$

$$= 2\sqrt{a^2 - b\, x^2}$$

38. $$\int \frac{dx}{(a + bx^2)^{\frac{3}{2}}} = \int \frac{dx}{b^{\frac{3}{2}} x^3 (ab^{-1} x^{-2} + 1)^{\frac{3}{2}}}$$

$$= \frac{1}{2ab^{\frac{1}{2}}} \int (ab^{-1} x^{-2} + 1)^{-\frac{3}{2}} d(ab^{-1} x^{-2} + 1)$$

$$= \frac{x}{a\sqrt{a + bx2}}.$$

39. $\int \frac{dx}{(a+bx)^{\frac{n+1}{n}}}$. El denominador se escribe:

$$[\, bx^n (ab^{-1} x^{-n} + 1)\,]^{\frac{n+1}{n}}$$

$$= -\frac{1}{nb^{\frac{1}{n}}} (ab^{-1} x^{-n} + 1)^{-\frac{n+1}{n}} d(ab^{-1} x^{-n} + 1);$$

resulta:

$$\int (a+bx)^{-\frac{n+1}{n}} dx = \frac{x}{a\sqrt[n]{a+bx^n}} \quad \text{(Roberts, 7)}$$

Esponenciales.

40. $\int_x e^{-x} = -\int d\, e^{-x} = -e^{-x} = -\frac{1}{e^x}$

41. $\int_x e^{ax} = \frac{1}{a} \int d\, e^{ax} = \frac{1}{a} e^{ax}$ (Perry, 281)

42 $\int_x e^{-ax} = -\frac{1}{a} \int d\, e^{-ax} = -\frac{1}{a} e^{-ax}$

43. $\int_x \frac{1}{1+e^x} = \int_x \frac{1}{e^x(1+e^{-x})} = L \frac{e^x}{1+e^{ax}}$

44. $\int_x \frac{e^{ax}}{1+e^{ax}} = \frac{1}{a} \int \frac{d\,(1+e^{ax})}{a+e^{ax}} = \frac{1}{a} L\,(1+e^{ax})$

45. $\int_x \frac{1}{\sqrt{e^{2x}-1}} = \frac{d\,x}{\sqrt{e^{2x}(1-e^{-2x})}} = -\int \frac{d\,e^{-2x}}{1-e^{-2x}}$

$= -\,\text{arc cos}\; e^{-x}$

46. $\int_x a^x = \frac{1}{L\,a} \int d\,a^x = \frac{a^x}{L\,a}$

47. $\int_x a^x = \frac{\log e}{\log a} \int \frac{\log a}{\log e} a^x\, d\,x = \frac{\log e}{\log a} a^x$ (Sonnet, 226)

48. $\int_x a^x$ Siendo $a^x = e^{xLa}$, tendremos:

$\int a^x\, d\,x = \int e^{xLa}\, d\,x = \frac{1}{L\,a} e^{xLa} = \frac{a^x}{Lx}$ (H. Cox, 27)

49. $\int_x a^{-x} = -\frac{\log e}{\log a} \int \frac{\log a}{\log e} a^x\, d\,x = -\frac{\log e}{\log a} a^{-x}$

(Pauly, 100)

50. $\int_x a^{a+bx} = \frac{1}{b\,L\,a} \int d\,a^{a+bx} = \frac{a^{a+bx}}{b\,L\,a}$

Logarítmica.

51. $$\int \frac{L\,x}{1+(L\,x)^2} = \frac{d\,x}{x}\;\frac{1}{2}\int \frac{d\,(L\,x)^2}{1+(L\,x)^2}$$

$$= \frac{1}{2}\,\mathrm{arc}^2\mathrm{tg}\,(L\,x)$$

Trigonométricas.

52. $$\int_x \mathrm{sen}\; m\,x = \frac{1}{m}\int \mathrm{sen}\; m\,x\, d\, m\, x = -\frac{1}{m}\cos m\,x$$

(Timmermans, 252)

53. $$\int_x \mathrm{sen}\;(00^o - x) = -\int \mathrm{sen}\;(90^o - x)\; d\,(90^o - x)$$

$$= -\left[-\cos\,(90^o - x)\right] = \mathrm{sen}\; x.$$

54. $$\int_x \mathrm{sen}\;(a\,\theta + b) = \frac{1}{2}\int \mathrm{sen}\;(a\,\theta + b)\; d\,(a\,\theta + b)$$

$$= -\frac{3}{2}\cos\,(a\,\theta + b)$$

55. $$\int_x x^{n-1}\,\mathrm{sen}\;(a + b\,x^n) = -\frac{1}{b}\cos\,(a + b\,x^n)$$

56. $$\int_x \cos a\,x = \frac{1}{a}\,\mathrm{sen}\; a\,x$$

57. $$\int_x ab\cos\,(b\,x + c) = \frac{a}{b}\,\mathrm{sen}\;(b\,x + c)$$

58. $\int_x \frac{\operatorname{sen} x}{\cos x} = -\int \frac{d \cos x}{\cos x} = -L \cos x = L \sec x.$

(Moigno, II, 39)

59. $\int_x \frac{\cos x}{\operatorname{sen}^2 x} = -\frac{1}{\operatorname{sen} x}$

60. $\int_x \frac{\cos x}{a + b \cos x} = -\frac{1}{b} \int \frac{d(a + b \cos x)}{a + b \cos x}$

$$= -\frac{1}{b} L(a + b \cos x)$$

61. $\int_x \frac{\operatorname{sen} \theta}{a - b \cos \theta} = \frac{1}{b} \int \frac{d\, b \cos \theta}{a - b \cos \theta}$

$$= \frac{1}{b} L(a - b \cos \theta)$$

62. $\int_x \frac{\cos x}{a^2 + b^2 \operatorname{sen} x} = \frac{1}{b^2} \int \frac{d\, b^2 \operatorname{sen} x}{a^2 + b^2 \operatorname{sen} x}$

$$= \frac{1}{b^2} L(a^2 + b^2 \operatorname{sen} x)$$ (Baahy, 24)

44. *Trasposicion de un término.*—Operacion correlativa a la trasposicion de la variable.

I. *Forma* $\frac{du}{1 - u^2} = L \sqrt{\frac{1 + x}{1 - x}}$ (Núm. 44, Ejer. 15)

1. $\int \frac{dx}{a^2 - x}$ Para reducir la fraccion a la forma indicada

sacamos factor comun a a^2 i en seguida descomponemos $\frac{1}{a^2}$ en $\frac{1}{a} \cdot \frac{1}{a}$.

$$\int \frac{dx}{a^2 - x^2} = \int \frac{dx}{a^2\left(1 - \frac{x^2}{a^2}\right)} = \frac{1}{a} \int \frac{\frac{1}{a}\,dx}{1 - \left(\frac{x}{a}\right)^2}$$

$$= \frac{1}{a} \int \frac{d\frac{x}{a}}{1 - \left(\frac{x}{a}\right)^2} = \frac{1}{a} \int \frac{du}{1 - u^2} = \frac{1}{a}\, L \sqrt{\frac{1 + \frac{x}{a}}{1 - \frac{x}{a}}}$$

o bien, despues de simplificar,

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\, L \sqrt{\frac{a + x}{a - x}}$$

La operacion anterior equivale a dividir los dos términos de la fraccion por a^2, i a trasponer uno de los factores:

$$\int \frac{\frac{dx}{a^2}}{1 - \frac{x^2}{a^2}} = \frac{1}{a} \int \frac{d\frac{x}{a}}{1 - \left(\frac{x}{a}\right)^2}$$

2. $$\int \frac{dx}{a - x^2} = \int \frac{dx}{a\left[1 - \left(\frac{x}{\sqrt{a}}\right)\right]^2} = \frac{1}{\sqrt{a}}\, L \left(\frac{\sqrt{a} + x}{\sqrt{a} - x}\right)^{\frac{1}{2}}$$

3. $$\int \frac{dx}{a - b x^2} = \frac{1}{2 b \sqrt{a}}\, L\, \frac{\sqrt{a} + \sqrt{b}\, x}{\sqrt{a} - \sqrt{b}\, x} \text{ (Peacock, 271)}$$

4. $\int \frac{dx}{x^2 - a^2} = -\int \frac{dx}{a^2 - x^2} = \frac{1}{a} L \sqrt{\frac{a-x}{a+x}}$

5. $\int \frac{dx}{a^2x\,2 - b^2} = -\frac{1}{a^2}\int \frac{dx}{\left(\frac{b}{a}\right)^2 - x^2} = \frac{1}{2a^2} L \frac{b-ax}{b+ax}$

6. $\int \frac{x\,dx}{a^2 b^2 - x^4} = \frac{1}{2}\int \frac{dx^2}{a^2 b^2 - (x^2)^2} = \frac{1}{4ab} L \frac{x^2+ab}{x^2-ab}$

(Brahy, 19)

II. *Forma.* $\frac{du}{1+u^2} = \text{arc tg } u.$

7. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\int \frac{d\frac{x}{a}}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a} \text{arc tg } \frac{x}{a}$

(Francoeur, 359)

8. $\int \frac{dx}{3+x^2} = \int \frac{dx}{3\left(1+\left(\frac{x}{\sqrt{3}}\right)^2\right)} = \frac{1}{\sqrt{3}}\int \frac{d\frac{x}{\sqrt{3}}}{1+\left(\frac{x}{\sqrt{3}}\right)^2}$

$= \frac{1}{\sqrt{3}} \text{arc tg } \frac{x}{\sqrt{3}}$

9. $\int \frac{dx}{a+bx^2} = \frac{1}{\sqrt{ab}} \text{arc tg} \sqrt{\frac{b}{a}}\, x$

$= \frac{1}{\sqrt{ab}} \text{arc cos} \sqrt{\frac{a}{a+bx^2}}$

$$= \frac{1}{\sqrt{a\,b}} \operatorname{arc\,sec} \sqrt{\frac{a + b\,x^2}{a}} \quad \text{(Perry, 28)}$$

10. $$\int \frac{d\,x}{2 + 3\,x^2} = \frac{1}{\sqrt{6}} \operatorname{arc\,tg} \sqrt{\frac{3}{2}}\,x$$

11. $$\int \frac{-\,d\,x}{a^2 + x^2} = \frac{1}{a} \operatorname{arc\,cot} \frac{x}{a}$$

12. $$\int_x \frac{1}{a^2 + b^2\,x^2} = \int_x \frac{1}{a^2\left[\,1 + \left(\frac{b\,x}{a}\right)^2\,\right]}$$

$$= \frac{1}{a\,b} \operatorname{arc\,tg} \frac{b}{a}x$$

13. $$\int_x \frac{x}{a^4 + x^4} = \frac{1}{2}\int_x \frac{2\,x}{(a^2)^2 + (x^2)^2} = \frac{1}{2\,a^2} \operatorname{arc\,tg} \frac{x^2}{a^2}$$

14. $$\int_x \frac{x}{a^2\,b^2 + x^4} = \frac{1}{2}\int_x \frac{d\,(x^2)}{a^2\,b^2 + (x^2)^2} = \frac{1}{2\,a\,b} \operatorname{arc\,tg} \frac{x^2}{a\,b}$$

III. *Forma.* $\frac{d\,u}{\sqrt{1 - u^2}} = \operatorname{arc\,sen} u$

15. $$\int \frac{d\,x}{\sqrt{a^2 - x^2}} = \int \frac{d\,x}{a\sqrt{1 - \left(\frac{x}{a}\right)}} = \int \frac{d\,\frac{x}{a}}{\sqrt{1 - \left(\frac{x}{a}\right)^2}}$$

$$= \operatorname{arc\,sen} \frac{x}{a}$$

La operacion anterior tambien se hace dividiendo por *a* los dos términos de la fraccion.

16. $$\int_x \frac{m}{\sqrt{a^2 - b z^2}} = \frac{m}{\sqrt{b}} \text{ arc (sen } \frac{3}{a} \sqrt{b}\,)$$

(Francoeur, 359)

17. $$\int_x \frac{1}{\sqrt{b^2 - c x^2}} = \frac{1}{\sqrt{c}} \text{ arc sen } \frac{\sqrt{c}}{b} x$$

18. $$\int \frac{a x}{\sqrt{b^4 - x^4}} = \frac{a}{2} \int \frac{d z}{\sqrt{(b^2)^2 - z^2}} = \frac{a}{2} \text{ arc sen } \frac{x^2}{b}$$

(Timmermans, 250)

19. $$\int_x \frac{\sqrt{3}}{\sqrt{5 - 3 x^2}} = \text{arc sen} \sqrt{\frac{3}{5}}\, x.$$

20. $$\frac{- d x}{\sqrt{a^2 - x^2}} = \text{arc } \frac{x}{a}$$

21. $$\frac{a^2 d x}{\sqrt{a^4 b^4 - x^4}} = \frac{1}{3} \int \frac{3 x^2}{\sqrt{a^4 b^4 \quad (x^3)^2}} d x$$

$$= \frac{1}{3} \text{ arc sen } \frac{x^3}{a^2 b^2}$$

IV *Forma*. $$\frac{d u}{\sqrt{2 u - u^2}} = \text{arc vers. } u$$

22. $$\int \frac{d x}{\sqrt{2 a x - x^2}} = \text{arc cos } \left(1 - \frac{x}{a}\right)$$

23. $$\int \frac{dx}{\sqrt{ax - bx^2}} = \int \frac{dx}{\sqrt{b}\sqrt{2 \cdot \frac{a}{2b} - x^2}}$$

$$= \frac{1}{\sqrt{b}} \text{ arc vers } \frac{2bx}{a}$$

V. *Forma.* $$\frac{du}{u\sqrt{u^2 - 1}} = \text{arc cosec } u$$

24. $$\int_x \frac{1}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \text{ arc cosec } \frac{x}{a}$$

VI. *Forma.* $$\frac{du}{\sqrt{u^2 \pm 1}} = L\,(x + \sqrt{x^2 \pm 1})$$

25. $$-\int \frac{x^{n-1}}{\sqrt{(ab)^{2m} + x^{2n}}}\, dx = \frac{1}{n} \int \frac{n x^{n-1}\, dx}{\sqrt{(ab)^{2m} + (x^n)^2}}$$

$$= \frac{1}{n} L \frac{x^n + \sqrt{(ab)^{2n} + x^{2n}}}{(ab)^m}$$

(Brahy, 20)

45. *Reducciones aljebraicas* o reducir un polinomio a un binomio.—Se emplean en la integracion de las diferenciales que contienen el trinomio de segundo grado $ax^2 + bx + c$.

Para darle la forma binomia, principiamos por sacar factor comun a *a*:

$$a\left(x^2 + \frac{b}{a} x + \frac{c}{a}\right);$$

completemos el cuadrado, agregando el cuadrado de la mitad del coeficiente de x,

$$\frac{b^2}{4a^2} - \frac{b^2}{4a^2} = 0:$$

$$a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \frac{c}{a} - \frac{b^2}{4a^2}\right]$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right];$$

representemos por u^2 el cuadrado i por m^2 el término cons tante; se obtiene:

$$ax^2 + bx + c = a(u^2 + m^2).$$

El binomio puede tener tres distintos valores, segun el signo de $4ac - b^2$, diferencia que puede ser positiva, nula o negativa.

1. $$\int \frac{dx}{a + bx + cx^2} = \frac{1}{c}\int \frac{dx}{\left(x + \frac{b}{2c}\right)^2 + \frac{4ac - b^2}{4c^2}}$$

$$= \frac{1}{c}\int \frac{du}{u^2 + m^2}.$$

Si m^2 es *positivo*, la integral es

$$\frac{2}{4ac - b^2}\ \text{arc tg}\ \frac{2cx + b}{\sqrt{4ac - b^2}};$$

i si es *negativo*

$$\frac{1}{\sqrt{b^2-4ac}}\,L\,\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2-4ac}}$$

(Todhunter, II, 14).

2. $$\int\frac{dx}{1+x+x^2}=\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{1}{\sqrt{3}}\operatorname{arc\,tg}\frac{2x+1}{\sqrt{3}}$$

3. $$\int\frac{dx}{1+x-x^2}=-\int\frac{dx}{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}}$$

$$=\frac{1}{\sqrt{5}}\,L\,\frac{2x-1+\sqrt{5}}{2x-1-\sqrt{5}}$$

4. $$\int\frac{dx}{3+2x+x2}=\int\frac{dx}{2+(1+x)^2}$$

$$=\frac{dz}{2+z^2}=\frac{1}{\sqrt{2}}\ \operatorname{arc\,tg}\ \frac{1+x}{\sqrt{2}}$$

5. $\int\frac{m+nx}{a+bx+cx^2}\,dx$. El numerador se escribe

$$\frac{n}{2c}\ (2cx+b)+m-\frac{nb}{2c}.$$

$$\therefore\ y=\frac{n}{2c}\int\frac{2cx+b}{a+bx+cx^2}\,dx+\left(m-\frac{nb}{2c}\right)\int\frac{dx}{a+bx+cx^2}.$$

La primera integral es:

$$\frac{n}{2c} L (a+bx+cx^2);$$

i la segunda se resuelve segun el ejercicio I. (Perry, 283)

6. $\int \frac{dx}{\sqrt{a+bx\pm cx^2}}$. Ateniéndonos al subradical R i al signo superior, encontramos:

$$R = \left(x + \frac{b}{2c}\right)^2 + \frac{4ac - b^2}{4c^2};$$

i al signo inferior:

$$R = -c\left(x^2 - \frac{bx}{2} - \frac{a}{c} = c\right)\left[\frac{b^2+4ac}{4c2} - \left(x - \frac{b}{2c}\right)^2\right];$$

de esta manera se reducen a las formas conocidas:

$$\frac{du}{\sqrt{u + a^2}}, \quad \frac{du}{\sqrt{a^2 - u^2}} \quad \text{(Gregory, 251)}$$

7. $$\int \frac{dx}{\sqrt{1+x+x^2}} = L\left(2x+1+2\sqrt{1+x+x^2}\right)$$

8. $$\int \frac{-dx}{\sqrt{2-5x-3x^2}} = \frac{1}{\sqrt{3}} \int \frac{-dx}{\sqrt{\frac{49}{36} - \left(x - \frac{5}{6}\right)^2}}$$

$$= \frac{1}{\sqrt{3}} \arccos \frac{6x+5}{7}$$

9. $$\int \frac{dx}{\sqrt{1+2x-x^2}} = \text{arc sen} \frac{x-1}{\sqrt{2}}$$

10. $$\int \frac{dx}{\sqrt{1+4x-x^2}} = \int \frac{dx}{\sqrt{5-(x-2)^2}} = \text{arc sen} \frac{x-2}{\sqrt{5}}$$

11. $$\int \frac{dx}{x\sqrt{a+bx+cx^2}} = \int \frac{dx}{x^2\sqrt{a x^{-2}+b x^{-1}+c}}$$

$$= -\int \frac{d(x^{-1})}{\sqrt{a x^{-2}+b x^{-1}+c}}$$

Si hacemos $x^{-1} = u \therefore u^2 = x^{-2}$, se reduce a la forma anterior:

$$\int \frac{du}{\sqrt{a u^2+b u+c}}$$

12. $$\int \frac{dx}{x\sqrt{1+x+x^2}} = L \frac{x}{2+x+2\sqrt{1+x+x^2}}$$

13. $$\int \frac{dx}{x\sqrt{4x^2-4x-1}} = \int \frac{x^{-2}\,dx}{\sqrt{4-4x^{-1}-x^{-2}}}$$

$$= \text{arc cos} \frac{2x+1}{2\sqrt{2}\,x} \text{ (Brahy, 9)}$$

14. $$\int \frac{dx}{(a+bx+cx^2)^{\frac{3}{2}}} = \frac{2(2cx+b)}{(4ac-b^2)(a+bx+cx^2)}$$

15. $$\int \frac{dx}{x^2 + px + q} = \int \frac{dx}{\left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)}$$

si $q > \frac{p^2}{4}$, la integral es de la forma arc tg u;

i si $q < \frac{p^2}{4}$, $\qquad y = \frac{1}{u^2 - m^2}$.

16. $$\int \frac{dx}{1 + 2(a+b)x - (a+b)^2 x^2} = \frac{1}{a+b} \text{ arc sen } \frac{(a+b)x - 1}{\sqrt{2}}$$

17. $$\frac{dx}{(x^2 + px + q)^{\frac{3}{2}}} = \frac{2}{4q - p^2} \cdot \frac{2x + p}{\sqrt{x^2 + px + q}}$$

18. $\int \frac{dx}{x^2 - 2ax}$. Aquí $x^2 - 2ax = x^2 - 2ax + a^2 - a^2$

$$= (x-a)^2 - a^2 = u^2 - a^2$$

$$\therefore \int \frac{dx}{x^2 - 2ax} = \frac{1}{2a} L \frac{x - 2a}{x + 2a}$$

19. $$\int \frac{dx}{\sqrt{2ax - x^2}} = -\int \frac{d(a-x)}{\sqrt{a^2 - (a-x)^2}}$$

$$= -\int \frac{d\,\frac{a-x}{a}}{1 - \left(\frac{a-x}{a}\right)^2} = \text{arc cos} \frac{x-a}{a}$$

20. $\int \frac{x\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \text{arc sen} \sqrt{\frac{x^2-a^2}{b^2-a^2}}$

46. *Reducciones trigonométricas.*—Para integrar una derivada trigonométrica, se reduce la funcion a *seno* o a otra funcion trigonométrica, i se aplica en seguida una fórmula conocida, tal como

$$\int -\text{sen}\, u\, d\, u = \cos u.$$

1. $\int_x \cos x = \int_x \text{sen}\,(90^o - x) = -\int \text{sen}\,(90^o - x)\, d\,(90^o - x)$

$$= \int -\text{sen}\, u\, du = \cos\,(90^o - x) = \text{sen}\, x.$$

(Peacock, 311)

2. $\int \text{tg}\, x\, d\, x = \int \frac{\text{sen}\, x}{\cos x}\, d\, x = -\int \frac{d \cos x}{\cos x}$

$$= - L \cos x = L \sec x$$ (Laurent, III, 61)

3. $\int \cot x\, d\, x = \int \frac{\cos x}{\text{sen}\, x}\, d\, x = \int \frac{d\, \text{sen}\, x}{\cos x} = L\, \text{sen}\, x$

$$= - L\, \text{cosec}\, x.$$

4. $\int \sec x\, d x$. *Primer método.* Reduzcamos a $\cos x$ i en seguida a $\text{sen}\, x = u$:

$$\int_x \sec x = \int_x \frac{1}{\cos x} = \int_x \frac{\cos x}{\cos^2 x} = \int \frac{d\, \text{sen}\, x}{1-\text{sen}^2\, x} = \int \frac{d\, u}{1-u^2}$$

Apliquemos la fórmula del número 45, ejercicio 1:

$$\int \sec x \, d x = L \sqrt{\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}}. \qquad \text{(A)}$$

Esta integral puede afectar diversas formas, porque siendo

$$\operatorname{sen} x = \cos (90^{o} - x) = -\cos (90^{o} + x),$$

tendremos:

$$\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x} = \frac{1+\cos (90^{o}-x)}{1-\cos (90^{o}-x)} = \cot \frac{1}{2} (90^{o}-x)$$

$$\therefore \int \sec x \, d x = L \cot \left(\frac{1}{4} \pi - \frac{1}{2} x \right). \qquad \text{(B)}$$

$$\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x} = \frac{1-\cos (90^{o}+x)}{1+\cos (90^{o}+x)} = \operatorname{tg} \frac{1}{2} (90^{o}+x)$$

$$\therefore \int \sec x \, d x = L \operatorname{tg} \left(\frac{1}{4} \pi + \frac{1}{2} x \right) \qquad \text{(C)}$$

$$\cot \left(45^{o} - \frac{1}{2} x \right) = \operatorname{tg} \left(45^{o} + \frac{1}{2} x \right) = \frac{1 + \operatorname{tg} \frac{1}{2} x}{1 - \operatorname{tg} \frac{1}{2} x}$$

$$\therefore \int \sec x \, d x = L \frac{1 + \operatorname{tg} \frac{1}{2} x}{1 - \operatorname{tg} \frac{1}{1} x} \qquad \text{(D)}$$

Segundo método. Multiplicamos por

$$\frac{\sec x+\operatorname{tg} x}{\sec x+\operatorname{tg} x}=1:$$

$$\int \sec x . \frac{\sec x+\operatorname{tg} x}{\sec x+\operatorname{tg} x}\, d\, x= \int \frac{\sec^2 x+\operatorname{tg} x \sec x}{\sec x+\operatorname{tg} x}\, d\, x$$

$$=\int \frac{d\,(\sec x+\operatorname{tg} x)}{\sec x+\operatorname{tg} x}=L\,(\sec x+\operatorname{tg} x) \quad \text{(Osborne, 181)}$$

$$\therefore \int_x \sec x = L\,(\sec x+\operatorname{tg} x) \qquad \text{(E)}$$

o bien

$$\int_x \sec x = L\, \frac{1+\operatorname{sen} x}{\cos x} \qquad \text{(F)}$$

Tercer método. Reducimos a cos x i despues a sen x:

$$\sec x\, d\, x=\frac{d\, x}{\cos x}=\frac{d\left(\frac{\pi}{2}+x\right)}{\operatorname{sen}\left(\frac{\pi}{2}+x\right)}=\frac{\operatorname{sen} u}{d\, u};$$

Reducimos u a $\frac{1}{2}u$ i luego dividimos por $\cos^2\frac{u}{2}$:

$$\frac{d\,u}{\operatorname{sen} u}=\frac{\frac{1}{2}\,d\,u}{\operatorname{sen}\frac{1}{2}u\cos\frac{1}{2}u}=\frac{\dfrac{d\,\frac{u}{2}}{\cos^2\frac{u}{2}}}{\operatorname{tg}\frac{u}{2}}=\frac{d\operatorname{tg}\frac{1}{2}u}{\operatorname{tg}\frac{1}{2}u}$$

$$=d\, L\operatorname{tg}\frac{1}{2}u$$

$$\therefore \int_x \sec x = L\,tg\,\frac{1}{2}\left(\frac{1}{2}\pi + x\right) \text{ (Moigno, II, 39)}$$

5. $$\int \text{cosec}\, x\, dx = \frac{dx}{\text{sen}\, x} = \int \frac{\text{sen}\, x\, dx}{\text{sen}^2 x} = -\int \frac{d\cos x}{1-\cos^2 x}$$

$$= -\, L \sqrt{\frac{1+\cos x}{1-\cos x}} = L \sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$\therefore \int_x \text{cosec}\, x = L\,tg\,\frac{1}{2}\,x;$$

o bien, aplicamos los dos últimos métodos anteriores.

6. $$\int \text{sen}\, x \cos x\, dx = \frac{1}{2}\int \text{sen}\, 2\, x\, dx = \frac{1}{4}\,\text{sen}\, 2x\, d\, 2x$$

$$= -\frac{1}{4}\cos 2\, x$$

(Serret, 77)

7. $$\int [\, 1 + tg^2 (a+b\, x)\,]\, d\, x = \int \sec^2 (a+b\, x)\, d\, x$$

$$= \frac{1}{b}\int d\; tg\, (a+b\, x) = \frac{1}{b}\, tg\, (a+b\, x) \qquad \text{(Cox, 39)}$$

Derivadas fraccionarias.

8. $$\int \frac{dx}{\text{sen}\, x} = \int \frac{dx}{2\,\text{sen}\, \frac{1}{2}x\, \cos \frac{1}{2}\, x} = L\, tg\, \frac{1}{2}\, x$$

9. $$\int \frac{dx}{\cos x} = \int \frac{d\left(\frac{1}{2}\pi + x\right)}{\operatorname{sen}\left(\frac{1}{2}\pi + x\right)} = L\,tg\left(\frac{1}{4}\pi + \frac{1}{2}x\right)$$

10. $$\int \frac{dx}{\operatorname{sen} x \cos x} \int \frac{\frac{dx}{\cos^2 x}}{\frac{\operatorname{sen} x}{\cos x}} = L\,tg\,x.$$

(Stoffaes, 121)

11. $$\int \frac{dx}{1-\cos x} = \int \frac{dx}{2\operatorname{sen}^2 \frac{1}{2}x}$$

$$= \int \operatorname{cosec}^2 \frac{1}{2}x\,d\,\frac{1}{2}x = -\cot \frac{1}{2}x$$

12. $$\int \frac{dx}{a(1+\cos x)} = \frac{1}{a}\int \frac{d\,\frac{1}{2}x}{\cos^2 \frac{1}{2}x} = \frac{1}{a}\int d\,tg\,\frac{1}{2}x$$

$$= \frac{1}{a}\,tg\,\frac{1}{2}x$$

13. $\int \frac{dx}{a + b\cos x}$ · Reduzcamos a $\frac{1}{2}x$:

$$a+b\cos x = a\left(\operatorname{sen}^2 \frac{1}{2}x + \cos^2 \frac{1}{2}x\right) + b\left(\cos^2 \frac{1}{2}x - \operatorname{sen}^2 \frac{1}{2}x\right)$$

$$= (a+b)\cos^2 \frac{1}{2}x + (a-b)\operatorname{sen}^2 \frac{1}{2}x$$

$$= \cos^2 \frac{1}{2} x \left[(a+b)+) + (a-b) \operatorname{tg}^2 \frac{1}{2} x \right]$$

$$\therefore \int \frac{dx}{a+b \cos x} = \int \frac{\sec^2 \frac{1}{2} x \, dx}{(a+b)+(a-b) \operatorname{tg}^2 \frac{1}{2} x}$$

$$= 2 \frac{d \operatorname{tg} \frac{1}{2} x}{(a+b)+(a-b) \operatorname{tg}^2 \frac{1}{2} x} .$$

si $a > b$, tendremos $y = \int \frac{du}{m+n u^2}$

$$= \frac{2}{\sqrt{a^2-b^2}} \operatorname{arc} \operatorname{tg} \left[\left(\frac{a-b}{a+b} \right)^{\frac{1}{2}} \operatorname{tg} \frac{1}{2} x \right]$$

si $a < b$: $y = \int \frac{du}{m - n u^2}$

$$= \frac{1}{\sqrt{b^2-a^2}} L \frac{\sqrt{b-a} \operatorname{tg} \frac{1}{2} x + \sqrt{b+a}}{\sqrt{b-a} \operatorname{tg} \frac{1}{2} x - \sqrt{b+a}}$$

(Grégory, 255)

14. $\int \frac{dx}{a+a \cos x} = \frac{1}{a} \operatorname{tg} \frac{1}{2} x$

51 $\int \frac{dx}{2+\cos x} = \frac{2}{\sqrt{3}} \operatorname{arc} \operatorname{tg} \frac{\operatorname{tg} \frac{1}{2} x}{\sqrt{3}}$

$$= \frac{1}{\sqrt{3}} \text{ arc cos } \frac{1+2\cos x}{d+\cos x}$$

16. $$\int \frac{d\,x}{1+2\cos x} = \frac{1}{\sqrt{3}} \text{ L } \frac{\sqrt{3}+\operatorname{tg}\frac{1}{2}x}{\sqrt{3}-\operatorname{tg}\frac{1}{2}x}$$

17. $$\int \frac{d\,x}{1+\cos^2 x} = \int \frac{d\,x}{\cos^2 x\,(1+\sec^2 x)} = \int \frac{\sec^2 d\,x}{2+\operatorname{tg}^2 x}$$

$$= \int \frac{d\,u}{2+u^2} = \frac{1}{\sqrt{2}} \int \frac{d\,\frac{u}{\sqrt{2}}}{1+\left(\frac{u}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}} \text{ arc tg } \frac{\operatorname{tg} x}{\sqrt{2}}$$

18. $$\int \frac{d\,x}{a+\cos^2 x} = \frac{d\,x}{\cos^2 x\,(1+a\sec^2 x)}$$

$$= \int \frac{d\operatorname{tg} x}{1+a+a\operatorname{tg}^2 x} = \int \frac{d\,u}{m+n\,u^2}$$

$$= \frac{1}{\sqrt{a^2+a}} \text{ arc tg } \left(\sqrt{\frac{a}{1+a}} \operatorname{tg} x\right).$$

19. $\int \frac{d\,x}{a+b\,\text{sen}\,x}$ · Se reduce a la forma anterior:

$$a+b\,\text{sen}\,x = a-b+b\left(\text{sen}^2\frac{1}{2}x+\cos^2\frac{1}{2}x\right)$$

$$+\ 2\,b\,\text{sen}\,\frac{1}{2}x\cos\frac{1}{2}x$$

$$= (a-b) + b\left(\operatorname{sen}\frac{1}{2}x + \cos\frac{1}{2}x\right)^2.$$

Pero $\operatorname{sen}\frac{1}{2}x + \cos\frac{1}{2}x = \operatorname{sen}\frac{1}{2}x + \operatorname{sen}\left(90^\circ - \frac{1}{2}x\right)$

$$= 2\operatorname{sen}45^\circ\cos\frac{1}{2}(x-90^\circ)$$

$$\therefore\ a+b\operatorname{sen}x = (a-b) + 2b\cos^2\left(\frac{1}{2}x - 45^\circ\right)$$

$$\therefore \int\frac{dx}{a+b\operatorname{sen}x} = \frac{1}{2b}\int\frac{dx}{\frac{a-b}{2b} + \cos^2\left(\frac{1}{2}x - 45^\circ\right)}$$

si $a > b$, se obtiene:

$$y = \frac{2}{\sqrt{a^2-b^2}}\operatorname{arc\,tg}\frac{a\operatorname{tg}\frac{1}{2}x + b}{\sqrt{a^2-b^2}}$$

i si $a < b$:

$$y = \frac{1}{\sqrt{b^2-a^2}}\operatorname{L}\frac{a\operatorname{tg}\frac{1}{2}x + b - \sqrt{b^2-a^2}}{a\operatorname{tg}\frac{1}{2}x + b + \sqrt{b^2-a^2}}$$

(Timmermans, 273)

20. $\int\frac{dx}{5+4\operatorname{sen}x} = \frac{2}{3}\operatorname{arc\,tg}\frac{1}{3}\left(5\operatorname{tg}\frac{1}{2}x + 4\right)$

21. $\int\frac{dx}{a+b\operatorname{tg}x}$. La fraccion se reduce a:

$$\frac{1}{a+b\,\dfrac{\operatorname{sen} x}{\cos x}}=\left(\frac{\cos x}{a\cos x+b\operatorname{sen} x}-\frac{a}{a^2+b^2}\right)+\frac{a}{a^2+b^2}$$

$$=\frac{b^2\cos x-ab\operatorname{sen} x}{(a^2+b^2)\,(a\cos x+b\operatorname{sen} x)}$$

$$=\frac{b}{a^2+b^2}\cdot\frac{b\cos x-a\operatorname{sen} x}{a\cos x+b\operatorname{sen} x}+\frac{a}{a^2+b^2}$$

$$\therefore\int\frac{dx}{a+b\operatorname{tg} x}=\frac{b}{a^2+b^2}\int\left(\frac{d\operatorname{sen} x+a\,d\cos x}{a\cos x+b\operatorname{sen} x}+\frac{a}{a^2+b^2}\right)dx$$

$$=\frac{1}{a^2+b^2}\left[\,b\,L\,(a\cos x+b\operatorname{sen} x)+a\,x\,\right]$$

(Brahy, 28)

22. $\dfrac{dx}{a\operatorname{sen} x+b\cos x}$ · *Primer método.* Hagamos

$$a=r\operatorname{sen}\alpha,\quad b=r\cos\alpha:$$

$$\int\frac{dx}{r\operatorname{sen}\alpha\operatorname{sen} x+r\cos\alpha\cos x}=\frac{1}{r}\int\frac{dx}{\cos(x-a)}$$

$$=\frac{1}{\sqrt{a^2+b^2}}\,L\operatorname{tg}\frac{x+a}{2}$$

(Serret. 79)

Segundo método. Introduzcamos $1=\dfrac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}$:

$$y = \frac{1}{\sqrt{a^2+b^2}} \int \frac{dx}{\frac{a \operatorname{sen} x}{\sqrt{a^2+b^2}} + \frac{b \cos x}{\sqrt{a^2+b^2}}}$$

Sea $\cos k = \frac{1}{\sqrt{a^2+b^2}} \therefore 1 - \cos^2 k = \frac{b^2}{a^2+b^2} = \operatorname{sen}^2 k$

$$\therefore y = \frac{1}{\sqrt{a^2+b^2}} \int \frac{dx}{\operatorname{sen}(x+k)} = L \operatorname{tg} \frac{x+k}{2}$$

$$L \left[\operatorname{tg} \frac{x}{2} + \frac{1}{2} \operatorname{arc} \cos \frac{a}{\sqrt{a^2+b^2}} \right]$$

(Sturm, 358)

23. $\int \frac{dx}{a \cos^2 x + b \operatorname{sen}^2 x} = \int \frac{dx}{\cos^2 x \, (a + b \operatorname{tg}^2 x)}$

$$= \frac{d \operatorname{tg} x}{a + b \operatorname{tg}^2 x} = \frac{1}{\sqrt{ab}} \operatorname{arc} \operatorname{tg} \left(\sqrt{\frac{b}{a}} \operatorname{tg} x \right)$$

24. $\int \frac{dx}{a \operatorname{sen} x + b \cos x + c}$.

Sea

$$a = r \operatorname{sen} \alpha; \; b = r \cos \alpha, \; \frac{c-r}{c+r} = \pm k^2 \therefore c = r \frac{1+k^2}{1-k^2}$$

$$\therefore y = \int \frac{dx}{r(\cos x - \alpha) + r \frac{1+k^2}{1-k^2}} = \frac{1}{r} \int \frac{dx}{\cos(x+\alpha) + \frac{1+k^2}{1-k^2}};$$

queda reducida al ejercicio 13.

(Tannery, II, 549)

25. $\int \frac{dx}{a\cos^2 x + b\,\mathrm{sen}^2 x + c}$ · Introduzcamos

$$\mathrm{sen}^2 x + \cos^2 x = 1 .$$

$$a\cos^2 x + b\,\mathrm{sen}^2 x + c\,\mathrm{sen}^2 x + c\cos^2 x$$

$$= (a+c)\cos^2 x + (b+c)\,\mathrm{sen}^2 x$$

$$= (a+c)\cos^2 x\,(1 + \frac{b+c}{a+c}\,\mathrm{tg}^2 x)$$

$$\therefore\ y = \frac{1}{a+c}\int \frac{d\,\mathrm{tg}\,x}{1 + \frac{b+c}{a+c}\,\mathrm{tg}^2 x} .$$

Integral de la forma $\int \frac{du}{1 + m u^2}$.

Derivadas irracionales.

26. $\int\sqrt{1+\cos x} = \int\sqrt{2}\,.\ \mathrm{sen}\ \frac{1}{2}\,x\,dx$

$$= 2\sqrt{2}\int \mathrm{sen}\ \frac{1}{2}\,x\ d\,\frac{1}{2}\,x$$

$$= 2\sqrt{2}\cos\frac{1}{2}\,x$$

27. $\int\sqrt{1-\cos x} = 2\sqrt{2}\,\mathrm{sen}\,\frac{1}{2}\,x$

28. $\int \frac{tg\ x\ d\ x}{\sqrt{a+b\ tg^2\ x}}$, aquí encontramos que

$$a+b\ tg^2\ x = a + b\ \frac{sen^2\ x}{cos^2\ x} = \frac{a\ cos^2\ x + b\ sen^2\ x}{cos^2\ x}$$

$$\therefore\ y = \int \frac{sen\ x\ d\ x}{\sqrt{a\ cos^2\ x + b\ sen^2 x}}$$

Por otra parte, $b\ sen^2 = b\ (1 - cos^2 x) = b - b\ cos^2 x$

$$\therefore\ a\ cos^2 x + b\ sen^2 x = b - (b-a)\ cos^2 x$$

$$\therefore\ y = -\frac{d\ cos\ x}{\sqrt{b-(b-a)\ cos^2 x}} = -\int \frac{d\ u}{\sqrt{1-u^2}} = arc\ cos\ u$$

$$= -\frac{1}{\sqrt{b-a}}\ arc\ cos \sqrt{\frac{b-a}{b}}\ cos\ x$$

(Grégory, 256)

CAPITULO IV

DESCOMPOSICIONES

47. *Definicion.*—Las descomposiciones se fundan en la regla del número 31, i consisten en descomponer el coeficiente diferencial en una suma de derivadas de fácil integracion.

48. *Desarrollo de un producto,* o efectuar las multiplicaciones indicadas.

1. $\int (a+bx)\,dx=\int a\,dx+\int bx\,dx=ax+\frac{1}{2}bx^2$

2. $\int (u\pm v\sqrt{-1})\,dx=\int u\,dx\pm\int v\sqrt{-1}\,dx$

(Moigno, II, 10)

3. $\int_x (Ax^m+Bx^n+Cx^p+\ldots)=\frac{Ax^{m+1}}{m+1}+\frac{Bx^{n+1}}{n+1}+\ldots$

(Hall, 221)

4. $\int_x\left(\frac{1}{3\sqrt[3]{x^2}}+\frac{1}{4\sqrt[4]{x^3}}=\frac{1}{5\sqrt[5]{x^4}}\right)$

$=\sqrt[3]{x}+\sqrt[4]{x}+\sqrt[5]{x}$

(Brahy, 19)

5. $\int_x (e^{mx}+e^{-mx})\,dx=\frac{1}{m}(e^{mx}-e^{-mx})$

(Sonnet, 227)

6. $\int (e^{m+x}-e^{-mx})\,dx=\frac{1}{m}(e^{mx}+e^{-mx})$

7. $\int (a-bx)\,x\,dx=\int_x ax-\int_x bx^2=\frac{1}{2}ax-\frac{1}{3}bx^3$

(Continuará)